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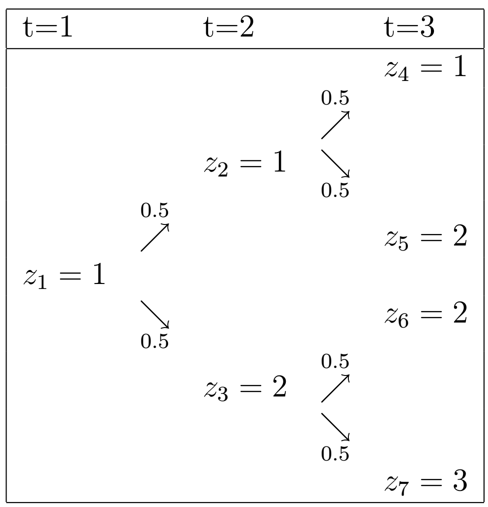
January 26, 2008

Exercise 1 *Exercise Final W03 Equilibrium and Uncertainty*

There are three dates t=1,2,3 and two households i=1,2. There is a single consumption good at each date and preferences are defined as $Eu(c_1, c_2, c_3)$ *.* $u()$ *is strictly concave, continuous and increasing. Households have one unit of time endowment in each period. The production technology is such that one unit of date t time can produce* z_t *units of date t consumption good. There is no storage. The value of* z_1 *is one, and* $z_t = z_{t-1}$ *with probability 0.5 and* $z_t = z_{t-1} + 1$ *with probability 0.5. Define the commodity space, the consumption sets, the utility functions, the production set, a Pareto-efficient allocation, a competitive equilibrium. Solve for equilibrium allocation and prices under the assumption of time-separable utility.*

First, we draw the tree and see that there are 7 periods/events. Hence there should be 7 contingencies of each commodity. Commodities are consumption and labor. So the commodity space should be 14-dimensional. $S = R^{2\times 7}$.



Consumption sets for household i=1,2:
$X_i = \{x \in S | x_j \geq 0, -1 \leq x_{j+7} \leq 0, j = 1...7\}$
Preferences for household i:
$U_i = \frac{1}{4}\left(u(x_1, x_2, x_4) + u(x_1, x_2, x_5) + u(x_1, x_3, x_6) + u(x_1, x_3, x_7)\right)$
Production set for the firm:
$Y = \{y \in S | y_j \leq -z_j y_{j+7}, y_{j+7} \leq 0, j = 1...7\}$
Resource constraint:
$X_1 + X_2 = Y$
Pareto optimal allocations are allocations (x^{1*}, x^{2*}, y^*) s.t.:
1) (x^{1*}, x^{2*}, y^*) is feasible: $x^{1*} \in X_1, \quad x^{2*} \in X_2, \quad y^* \in Y, \quad x^{1*} + x^{2*} = y^*$
2) $\nexists (x^{1\prime}, x^{2\prime}, y')$ s.t.

a)$(x^{1\prime}, x^{2\prime}, y')$ is feasible
b)$U_1(x^{1\prime}) \geq U_1(x^{1*})$ and $U_1(x^{2\prime}) \geq U_1(x^{2*})$
c)$U_1(x^{1\prime}) > U_1(x^{1*})$ or $U_1(x^{2\prime}) > U_1(x^{2*})$.

Assign θ_i the share of firm profits going to consumer i.

Competitive equilibria are allocations $(x^{1\star}, x^{2\star}, y^{\star})$ and prices $p^{\star} \in S$ s.t.

1) $(x^{1\star}, x^{2\star}, y^{\star})$ is feasible

2) $x^{i\star} = \arg\max_{x^i \in X_i} U_i(x^i) \,|p^{\star} x^i \leq \theta_i p^{\star} y^{\star}$

2) $y^{\star} = \arg\max_{y \in Y} p^{\star} y$

Separable utility implies a representation (horizon is finite, so assume no discouting):

$U(x) = u(x_1) + \frac{1}{2}(u(x_2) + u(x_3)) + \frac{1}{4}(u(x_4) + u(x_5) + u(x_6) + u(x_7))$ for each household

The pareto problem is:

$\lambda U(x^1) + (1-\lambda) U(x^2) \rightarrow \max_{x^i, y}$

s.t. $\quad x^1 + x^2 = y$ and feasibility

The resource constraints can be rewritten as:

$x_j^1 + x_j^2 = y_j \qquad x_j \geq 0, \qquad y_j \leq -z_j y_{j+7},$

$x_{j+7}^1 + x_{j+7}^2 = y_{j+7} \qquad -1 \leq x_{j+7} \leq 0, \qquad y_{j+7} \leq 0$

Notice, that since the utility does not depend on x_{j+7} the constraint $-1 \leq x_{j+7}$ has to be binding.

Therefore, $x_{j+7}^i = -1, y_{j+7} = -2$. Similarly, since utility is increasing in consumption, $y_j \leq 2z_j$ has to be binding to produce maximum output: $y_j = 2z_j$. For each subperiod the problem can be separately restated as:

$\lambda u\left(x_j^1\right) + (1-\lambda) u\left(x_j^2\right) \rightarrow \max$

$x_j^1 + x_j^2 = 2z_j$

$L = \lambda_1 u\left(x_j^1\right) + \lambda_2 u\left(x_j^2\right) + \mu\left(2z_j - x_j^1 - x_j^2\right)$

FOC$_i$: $\qquad \lambda_i u\left(x_j^i\right) = \mu$

Now since the consumers are identical a natural assumption would be to make pareto weights equal: $\lambda_1 = \lambda_2$. Then the planner would provide equal consumption to each of the agents:

$u\left(x_j^1\right) = \frac{\mu}{\lambda_1} = \frac{\mu}{\lambda_2} = u\left(x_j^2\right) \qquad \Rightarrow \qquad x_j^1 = x_j^2 = z_j.$

Summing up: a simmetric **Pareto-efficient allocation** is:

$x^{i*} = \{1, 1, 2, 1, 2, 2, 3, -1, -1, -1, -1, -1, -1, -1\}.$

$y^{*} = 2x^{i*} = \{2, 2, 4, 2, 4, 4, 6, -2, -2, -2, -2, -2, -2, -2\}.$

To find the prices supporting this allocation we need to solve each agent's problem:

$\Sigma_j \pi_j u(x_j) - \nu(\Sigma_j p_j x_j) \rightarrow \max_{x_j}$

FOC$_j$: $\quad \pi_j u'(x_j) = \nu p_j$

Normalizing $\nu = 1$ and substituting the optimal allocations we get the prices of the consumption good: $p_j = \pi_j u'(z_j)$.

Here the prices of labor do not enter the problem. For the prices of labor we should also solve the firm's problem:

$\Sigma_j p_j y_j + \Sigma_j p_{j+7} y_{j+7} + \phi \Sigma_j(-z_j y_{j+7} - y_j) \rightarrow \max_{y_j, y_{j+7}}$

FOC: $\quad p_j = \phi = \frac{p_{j+7}}{z_j}$

Therefore: $p_{j+7} = z_j p_j$ and the firm's profit is zero (as always with constant-returns-to-scale production functions).

Hence, the **equilibrium price vector** is:

$\{u'(1), \frac{1}{2}u'(1), \frac{1}{2}u'(2), \frac{1}{4}u'(1), \frac{1}{4}u'(2), \frac{1}{4}u'(2), \frac{1}{4}u'(3)...$

$u'(1), \frac{1}{2}u'(1), u'(2), \frac{1}{4}u'(1), \frac{1}{2}u'(2), \frac{1}{2}u'(2), \frac{3}{4}u'(3)\}$

Exercise 2 *Computing Private Information Problems*

There is a planner who hires an agent to operate a machine. The agent has reservation utility w_0. The agent exercises effort a, which is unobservable to the planner. Depending on effort, the probability distribution over output q varies. The probability that output q is realized given effort a is $p(q|a)$. We assume that this probability is positive for all combinations of q and a. Depending on the output q, the planner assigns consumption c to the agent, possibly at random. The agent derives utility $U(c, a)$ from consumption c and effort a. The planner is risk neutral and consumes $q - c$. To make computations possible, we assume that there are finite grids A, Q, and C for effort, output, and consumption. We can make the grids very fine to approximate a continuum of choices.

We are now going to formulate the planner's problem as a linear programming problem. The planner offers a contract $\pi(a, q, c)$ to the agent. Here $\pi(a, q, c)$ is the probability that the effort level is a, output is q, and the consumption of the agent is c. The a in this contract can be understood as a recommended effort. The planner cannot force the agent to put in a specific effort, since a is unobservable. The planner chooses a contract that maximizes expected consumption. The planner's problem is:

$$\max_{\pi} \Sigma_{A,Q,C} \pi\left(a, q, c\right)\left[q - c\right]$$

We assume that the planner has other sources of income, therefore we do not require the consumption of the planner to be nonnegative. The maximization is subject to a number of constraints. First, the contract has to be a probability distribution. We therefore require $\pi(a, q, c) \geq 0$ for all (a, q, c), and:

$$\Sigma_{A,Q,C} \pi\left(a, q, c\right) = 1$$

Next, we have to make sure that the contract satisfies the exogenously given probabilities $p(\bar{q}|\bar{a})$ of output $\bar{q}$ given effort $\bar{a}$. For all $\bar{a}$ and $\bar{q}$, we require:

$$\Sigma_{C} \pi\left(\bar{a}, \bar{q}, c\right) = p\left(\bar{q}|\bar{a}\right) \Sigma_{Q,C} \pi\left(\bar{a}, q, c\right) \qquad \Leftrightarrow \qquad \frac{\Sigma_{C} \pi(\bar{a}, \bar{q}, c)}{\Sigma_{Q,C} \pi(\bar{a}, q, c)} = p\left(\bar{q}|\bar{a}\right)$$

Actually, for each $\bar{a}$ one of these constraints is redundant. The agent has to be willing to work for the principal, instead of just receiving reservation utility. Therefore the contract has to deliver at least utility w_0 to the agent:

$$\Sigma_{A,Q,C} \pi\left(a, q, c\right) U\left(c, a\right) \geq w_0$$

Finally, we need incentive compatibility constraints to make sure that the agent actually takes the action that the planner recommends. For all a and $\hat{a}$, we require:

$$\Sigma_{Q,C} \pi\left(a, q, c\right) U\left(c, a\right) \geq \Sigma_{Q,C} \pi\left(a, q, c\right) \frac{p(q|\hat{a})}{p(q|a)} U\left(c, \hat{a}\right)$$

On the left-hand side is the expected utility the agent gets if he follows the recommended action. On the right-hand side is the utility the agent gets by taking action $\hat{a}$ instead of the recommended action a. This alternative action changes the probability distribution over output.

It is not hard to solve this problem for an arbitrary utility function and large grids for a, q, and c. However, for the purposes of this problem set we will use a simple example. The utility function is given by:

$$U(c, a) = \sqrt{c}\,(1 - a)$$

Effort a can take two values, $a = .2$ or $a = .4$. Output can take two values, $q = 1$ or $q = 2$. Consumption can take twenty values, evenly spaced between .1 and 2 (we assume that the consumer needs at least subsistence consumption .1). The conditional probability distributions over output are given by $p(q = 1|a = .2) = .8$, $p(q = 1|a = .4) = .2$, $p(q = 2|a = .2) = .2$, and $p(q = 2|a = .4) = .8$.

Compute solutions to the planner's problem for 20 different values of w_0, ranging from 0 to 1.1.

Repeat your computations for the full information case. That is, assume that the planner can observe effort. You need to compute the same program without the incentive constraints (4).

- What are the key differences between the constrained and the full-information solution?
- Conditional on effort and output, is the consumption of the agent randomized? Disregard randomizations that are caused by the finite grid only.
- Is the participation constraint binding for all w_0? Why or why not?

```
x=Linprog(f,A,b,Aeq,beq,lb,ub)
```

The routine solves the problem $\min_x f'x$ subject to $Ax \leq b$ and $Aeq\ x = beq$. f, b and beq are vectors, A is the matrix of inequality constraints, and Aeq defines the equality constraints. lb is a vector of lower bounds for x (use zeros), and ub is a vector of upper bounds (use ones). Notice that the routine minimizes the objective function, and the constraints are written as less-than-or-equal-to. You will have to write your program so that it fits this formulation.

The policy is a contract π proposed to the agent in period 1. He is also suggested the level of effort to take. In period 2 given the contract the agent decides whether he should take a different effort all other things being equal. In period 3 the output is created. In period 4 the firm flips a coin according to the policy and under the assumption that the agent told the truth and gives the agent some level of consumption. The incentive compartibility constraint is exactly the condition that the agent won't lie given the policy, other agents' actions and existence of commitment to it.

So we need to solve:

$\max_\pi \Sigma_{A,Q,C}\pi(a,q,c)[q-c]$

$\Sigma_{A,Q,C}\pi(a,q,c)=1$ Prob1

$0\le\pi(a,q,c)\le 1$ Prob2

$\Sigma_C\pi(\bar a,\bar q,c)=p(\bar q|\bar a)\Sigma_{Q,C}\pi(\bar a,q,c)$ Updating $\forall\bar a\forall\bar q$

$\Sigma_{A,Q,C}\pi(a,q,c)\sqrt{c}(1-a)\ge w_0$ Participation

$\Sigma_{Q,C}\pi(a,q,c)\sqrt{c}(1-a)\ge\Sigma_{Q,C}\pi(a,q,c)\frac{p(q|\hat a)}{p(q|a)}\sqrt{c}(1-\hat a)$ Truth-telling $\forall a\forall\hat a$

To solve you need to set up a grid for a,q,c and define a vector $x=\pi(a,q,c)$ on this grid. That's the solution you are looking for. In this case you would need $1\times 4M$ grid of the form:

$\left[\begin{array}{ccccccccc} q_1,a_1,c_1 & q_2,a_1,c_1 & q_1,a_2,c_1 & q_2,a_2,c_1 & ... & q_1,a_1,c_M & q_2,a_1,c_M & q_1,a_2,c_M & q_2,a_2,c_M\end{array}\right]$

Let $p(\bar q|\bar a)$ be given by: $\left[p_{q_1|a_1},p_{q_2|a_1},p_{q_1|a_2},p_{q_2|a_2}\right]$.

Then the function you need to minimize is:

$\max_\pi \Sigma_{A,Q,C}\pi(a,q,c)[q-c] \quad\Leftrightarrow$

$-\left[\begin{array}{ccccccccc} q_1-c_1 & q_2-c_1 & q_1-c_1 & q_2-c_1 & ... & q_1-c_M & q_2-c_M & q_1-c_M & q_2-c_M\end{array}\right]x\to\min$

$0\le x_i\le 1$

$\Sigma_{A,Q,C}\pi(a,q,c)=1 \quad\Leftrightarrow\quad \left[\begin{array}{ccccc}1 & 1 & 1 & 1 & ...\end{array}\right]x=1$

$\Sigma_C(\pi(\bar a,\bar q,c)-p(\bar q|\bar a)\Sigma_Q\pi(\bar a,q,c))=0 \quad\Leftrightarrow$

$\Sigma_C(\pi(\bar a,\bar q,c)-p(\bar q|\bar a)(\pi(\bar a,q_1,c)+\pi(\bar a,q_2,c)))=0 \quad\Leftrightarrow$

$\Sigma_C((1-p(\bar q|\bar a))\pi(\bar a,\bar q,c)-p(\bar q|\bar a)\pi(\bar a,\bar q_{not},c))=0$ for each $\bar a$ for each $\bar q$

$\left[\begin{array}{ccccccccc}1-p_{q_1|a_1} & -p_{q_1|a_1} & 0 & 0 & ... & 1-p_{q_1|a_1} & -p_{q_1|a_1} & 0 & 0\end{array}\right]x=0$ for q_1,a_1

$\left[\begin{array}{ccccccccc}-p_{q_2|a_1} & 1-p_{q_2|a_1} & 0 & 0 & ... & -p_{q_2|a_1} & 1-p_{q_2|a_1} & 0 & 0\end{array}\right]x=0$ for q_2,a_1

$\left[\begin{array}{ccccccccc}0 & 0 & 1-p_{q_1|a_2} & -p_{q_1|a_2} & ... & 0 & 0 & 1-p_{q_1|a_2} & -p_{q_1|a_2}\end{array}\right]x=0$ for q_1,a_2

$\left[\begin{array}{ccccccccc}0 & 0 & -p_{q_2|a_2} & 1-p_{q_2|a_2} & ... & 0 & 0 & -p_{q_2|a_2} & 1-p_{q_2|a_2}\end{array}\right]x=0$ for q_2,a_2

$\Sigma_{A,Q,C}\pi(a,q,c)\sqrt{c}(1-a)\ge w_0$

$-\left[\begin{array}{ccccc}(1-a_1)\sqrt{c_1} & (1-a_1)\sqrt{c_1} & (1-a_2)\sqrt{c_1} & (1-a_2)\sqrt{c_1} & ...\end{array}\right]x\le -w_0$

$\Sigma_{Q,C}\pi(\bar a,q,c)\left[\sqrt{c}(1-\bar a)-\frac{p(q|\hat a)}{p(q|\bar a)}\sqrt{c}(1-\hat a)\right]\ge 0$ for each $\bar a,\hat a$

$\frac{p(q|\hat a)}{p(q|\bar a)}=\left[\frac{p_{q_1|a_2}}{p_{q_1|a_1}},\frac{p_{q_2|a_2}}{p_{q_2|a_1}},\frac{p_{q_1|a_1}}{p_{q_1|a_2}},\frac{p_{q_2|a_1}}{p_{q_2|a_2}}\right]$

$-\left[\begin{array}{ccccc}(1-a_1)\sqrt{c_1}-\frac{p_{q_1|a_2}}{p_{q_1|a_1}}(1-a_2)\sqrt{c_1} & (1-a_1)\sqrt{c_1}-\frac{p_{q_2|a_2}}{p_{q_2|a_1}}(1-a_2)\sqrt{c_1} & 0 & 0 & ...\end{array}\right]x\le 0$ for a_1

$-\left[\begin{array}{ccccc}0 & 0 & (1-a_2)\sqrt{c_1}-\frac{p_{q_1|a_1}}{p_{q_1|a_2}}(1-a_1)\sqrt{c_1} & (1-a_2)\sqrt{c_1}-\frac{p_{q_2|a_1}}{p_{q_2|a_2}}(1-a_1)\sqrt{c_1} & ...\end{array}\right]x\le 0$ for a_2

So then

$f=-\left[\begin{array}{ccccc}q_1-c_1 & q_2-c_1 & q_1-c_1 & q_2-c_1 & ...\end{array}\right]$

$$A=-\left[\begin{array}{c}(1-a_1)\sqrt{c_1}-\frac{p_{q_1|a_2}}{p_{q_1|a_1}}(1-a_2)\sqrt{c_1},\quad (1-a_1)\sqrt{c_1}-\frac{p_{q_2|a_2}}{p_{q_2|a_1}}(1-a_2)\sqrt{c_1},\quad 0,\quad 0,\quad ...\\ 0,\quad 0,\quad (1-a_2)\sqrt{c_1}-\frac{p_{q_1|a_1}}{p_{q_1|a_2}}(1-a_1)\sqrt{c_1},\quad (1-a_2)\sqrt{c_1}-\frac{p_{q_2|a_1}}{p_{q_2|a_2}}(1-a_1)\sqrt{c_1},\quad ...\\ (1-a_1)\sqrt{c_1},\quad (1-a_1)\sqrt{c_1},\quad (1-a_2)\sqrt{c_1},\quad (1-a_2)\sqrt{c_1},\quad ...\end{array}\right]$$

$$b=\left[\begin{array}{c}0\\0\\-w_0\end{array}\right]\qquad Aeq=\left[\begin{array}{ccccc}1-p_{q_1|a_1} & -p_{q_1|a_1} & 0 & 0 & ...\\ -p_{q_2|a_1} & 1-p_{q_2|a_1} & 0 & 0 & ...\\ 0 & 0 & 1-p_{q_1|a_2} & -p_{q_1|a_2} & ...\\ 0 & 0 & -p_{q_2|a_2} & 1-p_{q_2|a_2} & ...\\ 1 & 1 & 1 & 1 & ...\end{array}\right]\qquad beq=\left[\begin{array}{c}0\\0\\0\\0\\1\end{array}\right]$$

$lb = [\ 0\quad 0\quad 0\quad 0\quad ...\ ]$ $ub = [\ 1\quad 1\quad 1\quad 1\quad ...\ ]$

Construct these matrices and call: x0=linprog(-f,A,b,Aeq,beq,lb,ub)

EXAMPLE CODE:

```
MAX=20;
                                        %define values
c=(1:MAX)*2/MAX;            a=[0.2 0.4];            q=[1 2];
P=[0.8 0.2 0.2 0.8];         w0=((1:MAX)-1)*1.1/(MAX-1);
                                        %define indicator matrices and payoffs
U=zeros(1,MAX*4);           Ij=zeros(2,MAX*4);           Ik=zeros(2,MAX*4);
Il=zeros(4,MAX*4);          f=zeros(MAX*4,1);
for i=1:MAX
    for j=1:2
        for k=1:2
            l=2*(j-1)+k;
            m=4*(i-1)+l;
            U(m)=sqrt(c(i))*(1-a(j));
            f(m)=q(k)-c(i);
        if j==1
            Ij(1,m)=1;
        elseif j==2
            Ij(2,m)=1;
        end
        if k==1
            Ik(1,m)=1;
        elseif k==2
            Ik(2,m)=1;
        end
        if l==1
            Il(1,m)=1;
        elseif l==2
            Il(2,m)=1;
        elseif l==3
            Il(3,m)=1;
        elseif l==4
            Il(4,m)=1;
        end
        end
    end
end
%define matrices for constraints
Aeq=[zeros(4,4*MAX);ones(1,4*MAX)];          A=[zeros(2,4*MAX);-U];
for i=1:MAX
    for j=1:2
        for k=1:2
            l=2*(j-1)+k;
```

```
                m=4*(i-1)+l;
                Aeq(1,m)=(P(1)-Ik(1,m))*Ij(1,m);
                Aeq(2,m)=(P(2)-Ik(2,m))*Ij(1,m);
                Aeq(3,m)=(P(3)-Ik(1,m))*Ij(2,m);
                Aeq(4,m)=(P(4)-Ik(2,m))*Ij(2,m);
                A(1,m)=((P((3-j-1)*2+k)*U(m+6-4*j)/P((j-1)*2+k))-U(m))*Ij(1,m);
                A(2,m)=((P((3-j-1)*2+k)*U(m+6-4*j)/P((j-1)*2+k))-U(m))*Ij(2,m);
            end
        end
    end
    lb=zeros(4*MAX,1);        ub=ones(4*MAX,1);     beq=[0;0;0;0;1];
    %have IC constraints
    x=zeros(MAX,4*MAX);
    for h=1:MAX
        b=[0;0;-w0(h)];
        x0=linprog(-f,A,b,Aeq,beq,lb,ub);
        x(h,:)=x0';
    end
    % no IC constraints
    y=zeros(MAX,4*MAX);
    for h=1:MAX
        b=[-w0(h)];
        y0=linprog(-f,A(3,:),b,Aeq,beq,lb,ub);
        y(h,:)=y0';
    end
```

When outside opportunities are low the firm wants all workers to work hard. In the case of full information it does not pay anything for high level of effort. For the private information case the firm has to pay workers some minimum amount to encourage them not to lie. Workers receive their information rent. This divergence of payments for different output among high-effort workers remains for all levels of outside opportunities just to encourage them to tell the truth. There is no divergence in payments among low-effort workers.

As outside opportunities increase the payments rise. At the level of outside utility of about 0.6 some workers are given an advice to be lazy. At the level of 0.8 most workers are lazy. There is no place for high effort at the level of 1. This happens because the firm has to encourage the workers to participate at all, since this gives the firm at least output of 1. In this case the firm just cannot afford high levels of efforts.

Insurance problem

Risk-neutral planner, risk-averse agent.

$\max \Sigma_{t=0}^{\infty} \beta^t \Sigma_{s^t} p(s^t)(-\tau(s^t))$ planner's

$\Sigma_{t=0}^{\infty} \beta^t \Sigma_{s^t} p(s^t) u(y(s^t) + \tau(s^t)) = v$ promise-keeping constraint

$\Sigma_{t=0}^{\infty} \beta^t \Sigma_{s^t} p(s^t|s^k) u(y(s^t) + \tau(s^t)) \geq \Sigma_{t=0}^{\infty} \beta^t \Sigma_{s^t} p(s^t|s^k) u(y(s^t) + \tau(\hat{\sigma}(s^t))) \quad \forall s^k$ truth-telling constraint

Redefine variables:

$w(s^k) = \Sigma_{t=k}^{\infty} \beta^{t-k} \Sigma_{s^t} p(s^t|s^k) u(y(s^t) + \tau(s^t))$

$P(v)$ is the maximized surplus of the planner for any given reservation utility v

Recursively:

$P(v) = \Sigma_{t=0}^{\infty} \beta^t \Sigma_{s^t} p(s^t)(-\tau(s^t)) = \Sigma_{t=0}^{\infty} \beta^t \Sigma_{s^t} p(s^t|s^0)(-\tau(s^t))$

$= -\tau(s^0) + \beta \Sigma_{t=1}^{\infty} \beta^{t-1} \Sigma_{s^t} p(s^t|s^0)(-\tau(s^t)) =$

$= -\tau(s^0) + \beta \Sigma_{s^1} p(s^1|s^0) \left\{ \Sigma_{t=1}^{\infty} \beta^{t-1} \Sigma_{s^t} p(s^t|s^1)(-\tau(s^t)) \right\} =$

$= -\tau(s^0) + \beta \Sigma_{s^1} p(s^1|s^0) P(v_1)$

Recursive formulation:

$P(v) = \max_{\tau^n, w^n} \Sigma_{n=1}^{N} \pi_n [-\tau^n + \beta P(w^n)]$

$\Sigma_{n=1}^{N} \pi_n [u(y^n + \tau^n) + \beta w^n] = v$

$C_{n,m} = u(y^n + \tau^n) + \beta w^n - u(y^n + \tau^m) + \beta w^m \geq 0$

Second is an equality because the planner would violate his previous resource constraint.

1) Bounds:

$\tau^n \in [\inf c - y^n, \infty] \qquad w^n \in \left[-\infty, \sup \frac{u(c)}{1-\beta}\right]$

If a contract pays b in each period, then the discounted profits are $\frac{-b}{1-\beta}$, and that's the lower bound on $P(v)$ given that $v = \Sigma_{n=1}^{N} \pi_n \frac{u(y^n+b)}{1-\beta}$. That's the lowest profit a planner can get, while delivering v.First-best is when the planner can perfectly insure: $v = \Sigma_{n=1}^{N} \pi_n \frac{u(c)}{1-\beta}$.

Therefore: $-\frac{b(v)}{1-\beta} \leq P(v) \leq \Sigma_{n=1}^{N} \pi_n \frac{y^n - c(v)}{1-\beta}$.

2) Since $\lim_{c \to \inf c} u'(c) = \infty \quad \Rightarrow \quad \lim_{v \to -\infty} P'(v) = 0$

Since $\lim_{c \to \infty} u'(c) = 0 \quad \Rightarrow \quad \lim_{v \to \sup \frac{u(c)}{1-\beta}} P'(v) = -\infty \qquad \lim_{v \to \sup \frac{u(c)}{1-\beta}} P(v) = -\infty$

Let $y^n < y^m$

$u(y^n + \tau^n) + \beta w^n \geq u(y^n + \tau^m) + \beta w^m$

$u(y^m + \tau^m) + \beta w^m \geq u(y^m + \tau^n) + \beta w^n$

Hence, $u(y^m + \tau^m) + u(y^n + \tau^n) \geq u(y^m + \tau^n) + u(y^n + \tau^m)$

$u(y^m + \tau^m) - u(y^n + \tau^m) \geq u(y^m + \tau^n) - u(y^n + \tau^n)$

If $y^m \geq y^n$ then $\tau^m \leq \tau^n$ and also $w^m \geq w^n$

Other results:

3) if $-\frac{u''}{u'}$ is nonincreasing, then $P(v)$ is concave.

4) Local constraints are enough: if $C_{n,n-1} \geq 0$ and $C_{n,n+1} \geq 0$ then all $C_{n,s} \geq 0$.

5) Local downward constraints bind: $C_{n,n-1} = 0$.

6) Both houshold utility and planner's profits increase with a higher income realization.

7) Martingale property of marginal utility: $P'(v_t) = E_t P'(v_{t+1}) \quad \Leftrightarrow \quad P'(v) = \Sigma_n \pi_n P'(w_n)$

It's a nonpositive martingale $\Rightarrow P'(v)$ converges almost surely to 0. That implies v converges almost surely to $-\infty$.

If that was not the case then w_n wouldn't be spread out, and there would be no incentive compartibility.