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Exercise 1 *Teamwork in General Equilibrium (Midterm 04W)*

The economy consists of two types of agents: measure μ_S skilled and μ_U unskilled with skill levels S>U>0. Output can only be produced in teams of two people using a production function $f(z_i, z_j) = z_i z_j$ for each team. There are three possible types of teams: $\{SS, SU, UU\}$. The utility of each worker is defined as: $u(c) + v(m)$ where $m \in \{0, S, U\}$ corresponds to the type of the teammate or 0 if the worker does not work.

a) Choose a suitable commodity space for this economy. Define consumption sets, preferences, the aggregate production set and the resource constraint.

b) Formulate the social planning problem for this economy.

c) If people don't care about their teammate and whether they work or not: $v(0) = v(S) = v(U)$. What can you say about the types of teams that will arise in the social optimum. Solve for the socially optimal allocation.

Solution

All the workers will be given a lottery, with possibilities of not working, matching an U or S worker. So each worker will choose consumption and two probabilities. We shall allow for the lotteries of the two agents to be different depending on their types. Hence the consumption space consists of 5 numbers (consumption and two probabilities for each type).

a) $S = R^5 = \{c, \pi_{U,U}, \pi_{U,S}, \pi_{S,U}, \pi_{S,S}\}$

In the consumption set $\pi_{i,j}$ denotes probability of type i to match with type j.

$X_U = \{x_U \in S | x_{Ui} \geq 0, x_{U2} + x_{U3} \leq 1, x_{U4} = x_{U5} = 0\}$

$U_U(x_U) = u(x_{U1}) + (1 - x_{U2} - x_{U3}) v(0) + x_{U2} v(U) + x_{U3} v(S)$

$X_S = \{x_S \in S | x_{iS} \geq 0, x_{S2} = x_{S3} = 0, x_{S4} + x_{S5} \leq 1\}$

$U_U(x_S) = u(x_{S1}) + (1 - x_{S4} - x_{S5}) v(0) + x_{S4} v(U) + x_{S5} v(S)$

In the production set $\pi_{i,j}$ denotes total number of people type i having matched with type j.

$Y = \left\{y \in S | y_i \geq 0, y_1 \leq \frac{y_2}{2} U^2 + \frac{y_5}{2} S^2 + \frac{y_3 + y_4}{2} US, y_4 = y_3\right\}$

$RC: \qquad \mu_U x_U + \mu_S x_S = y$

b) The social planner maximizes the ex ante utility of each worker:

$W = \mu_U U_U(x_U) + \mu_S U_S(x_S)$

c) If $v(x) = 0$ then $W = \mu_U u(c_U) + \mu_S u(c_S) \to \max$

s.t. $y_1 \leq \frac{y_2}{2} U^2 + \frac{y_5}{2} S^2 + y_3 US$

s.t. $\mu_U c_U + \mu_S c_S = y_1$ s.t. $\mu_U \pi_2 = y_2$ s.t. $\mu_U \pi_3 = y_3$

s.t. $\mu_S \pi_4 = y_3$ s.t. $\mu_S \pi_5 = y_5$ s.t. $\pi_2 + \pi_3 \leq 1$ s.t. $\pi_4 + \pi_5 \leq 1$

By substituting we can simplify to get:

$$\begin{cases} \mu_U u(c_U) + \mu_S u(c_S) \to \max \\ \mu_U c_U + \mu_S c_S \leq \frac{y_2}{2} U^2 + \frac{y_5}{2} S^2 + y_3 US \\ y_2 + y_3 \leq \mu_U, \qquad y_3 + y_5 \leq \mu_S \end{cases}$$

Assume all hold with equality:

$y_2 = \mu_U - y_3, \qquad y_5 = \mu_S - y_3$

$\mu_U c_U + \mu_S c_S = \frac{\mu_U - y_3}{2} U^2 + \frac{\mu_S - y_3}{2} S^2 + y_3 US = \mu_U \frac{U^2}{2} + \mu_S \frac{S^2}{2} - y_3 \frac{(U-S)^2}{2}$

To get the highest possible output we set $y_3 = 0$. It doesn't enter anywhere else. That means, there will be no mixed teams.

$\mu_U u(c_U) + \mu_S u(c_S) \to \max \qquad \text{s.t.} \qquad \mu_U c_U + \mu_S c_S = \mu_U \frac{U^2}{2} + \mu_S \frac{S^2}{2} = C$
$L = \mu_U u(c_U) + \mu_S u(c_S) + \lambda (C - \mu_U c_U - \mu_S c_S) \to \max_{c_S, c_U}$

FOC: $\quad u'(c_U) = \lambda = u'(c_S) \qquad$ Therefore, $c_U = c_S = \frac{\mu_U \frac{U^2}{2} + \mu_S \frac{S^2}{2}}{\mu_U + \mu_S} = \frac{C}{\mu_U + \mu_S} = c$

Social planning allocation:

$x_U = \{c, 1, 0, 0, 0\} \qquad x_S = \{c, 0, 0, 0, 1\} \qquad y = \{C, \mu_U, 0, 0, \mu_S\}.$

Exercise 2 *Mechanism Design*

There is a measure one of ex ante identical consumers. First all the decisions are made, then uncertainty is realized. With probability $p = \frac{1}{2}$ each consumer becomes type I and with probability $(1-p) = \frac{1}{2}$ she becomes type II. Type I consumers are risk-averse: $u_1(c) = \theta_1 \sqrt{c}$, while type II consumers are risk-neutral: $u_2(c) = \theta_2 c$. The endowment is $e = \frac{1}{2}$ per person. The planner can allocate resources (make transfers) but cannot see the types. First types become known to the consumers, then the planner asks the consumers about their types, and using the message assigns each consumer an allocation. Write the planner's problem. Use the revelation principle to find the optimal truth-telling mechanism. (For computational purposes assume $\theta_2 = \theta_1, c_1 = 0$). Is this the first-best allocation?

Planner's problem (general version): define a grid $\{c_1, c_2, ..., c_n\}$

$\max_{x_n^i} \Sigma_i p_i \Sigma_n x_n^i u_i(c_n)$

s.t. $\quad x_n^i \geq 0, \quad \Sigma_n x_n^i = 1.\ \forall i \quad$ (x_n^i is a distribution over c_n)

s.t. $\quad \Sigma_i p_i \Sigma_n x_n^i c_n \leq e \quad \forall i \quad$ (resource constraint)

s.t. $\quad \Sigma_n x_n^i u_i(c_n) \geq \Sigma_n x_n^j u_i(c_n) \quad \forall i, \forall j \quad$ (incentive compartibility constraint)

s.t.* $\quad \Sigma_i p_i \Sigma_n x_n^i u_i(c_n) \geq u_0 \quad$ (participation constraint)

Here u_0 is an outside option, which can constrain the planner even more, if present. It guarantees that ex ante each participant is better off participating in the game.

a) In our case $i = \{I, II\}$. As described in the lecture, risk-aversion of consumer 2 can be used to reveal his type. So a lottery would be given to type II (to which he is indifferent) while type I will try to avoid it and get a fixed amount. Therefore, we can use just three consumption points (which we don't know yet). Let them be $\{c_1, c_2, c_3\}$. As usual, consumption is non-negative.

$p\theta_1 \sqrt{c_2} + (1-p)\theta_2((1-\pi)c_1 + \pi c_3) \to \max_{\pi \in [0,1], 0 \leq c_1 \leq c_2 \leq c_3}$

s.t. $\quad pc_2 + (1-p)((1-\pi)c_1 + \pi c_3) \leq e \quad$ (resource constraint)

s.t. $\quad \theta_1 \sqrt{c_2} \geq (1-\pi)\theta_1 \sqrt{c_1} + \pi\theta_1 \sqrt{c_3} \quad$ (incentive compatibility for type I)

s.t. $\quad (1-\pi)\theta_2 c_1 + \pi\theta_2 c_3 \geq \theta_2 c_2 \quad$ (incentive compatibility for type II)

b) If IC_1 holds as equality and the grid points do not coincide, then IC_2 holds as an inequality:

$\sqrt{c_2} = (1-\pi)\sqrt{c_1} + \pi\sqrt{c_3} \qquad \Rightarrow \qquad (1-\pi)c_1 + \pi c_3 > c_2$

Hence IC_2 is not binding. Assume additionally that $c_1 = 0$ (we would want it to be as small as possible). Then the problem is simplified to:

$p\theta_1 \sqrt{c_2} + (1-p)\theta_2 \pi c_3 \to \max_{\pi \in [0,1], 0 < c_2 < c_3}$

s.t. $\quad pc_2 + (1-p)\pi c_3 = e \quad$ (RC)

s.t. $\quad \sqrt{c_2} = \pi\sqrt{c_3} \quad$ (IC_1)

Simplify: $\quad c_2 = \pi^2 c_3 \qquad p\pi^2 c_3 + (1-p)\pi c_3 = e$

$c_3 = \frac{e}{p\pi + 1 - p}\frac{1}{\pi} \qquad c_2 = \pi^2 c_3 = \frac{e\pi}{p\pi + 1 - p}$

Now set $p = \frac{1}{2}$

$\theta_1 \sqrt{\frac{2e\pi}{\pi+1}} + \theta_2 \frac{2e}{\pi+1} \to \max_\pi$

Derivatives: $\quad \frac{\partial}{\partial\pi}\sqrt{\frac{\pi}{\pi+1}} = \frac{1}{2\sqrt{\pi(\pi+1)}(\pi+1)} \qquad \frac{\partial}{\partial\pi}\frac{1}{\pi+1} = -\frac{1}{(\pi+1)^2}$

FOC: $\theta_1\sqrt{2e}\frac{1}{2\sqrt{\pi(\pi+1)}(\pi+1)} = \theta_2\frac{2e}{(\pi+1)^2}$

Simplifies to $\frac{\sqrt{\pi+1}}{\sqrt{\pi}} = \frac{\theta_2}{\theta_1}\sqrt{8e} = a \qquad \Rightarrow \qquad \pi = \frac{1}{a^2-1} = \frac{1}{8e\left(\frac{\theta_2}{\theta_1}\right)^2-1}$

Let $\theta_2 = \theta_1$, then if $a^2 = 8e \geq 2 \Leftrightarrow e \geq \frac{1}{4}$, then $\pi = \frac{1}{8e-1} \leq 1$.

In our case $e = \frac{1}{2}$, hence, $\pi = \frac{1}{3}$, $\quad c_3 = \frac{1}{\pi+1}\frac{1}{\pi} = \frac{9}{4} \quad c_2 = \frac{\pi}{\pi+1} = \frac{1}{4}$.

All constraints hold:

RC: $\frac{1}{2}\frac{1}{4} + \frac{1}{2}\frac{1}{3}\frac{9}{4} = \frac{1}{2}$ $\quad$ IC$_1$: $\sqrt{\frac{1}{4}} = \frac{1}{3}\sqrt{\frac{9}{4}} + \frac{2}{3}0$ $\quad$ IC$_2$: $\frac{1}{3}\frac{9}{4} > \frac{1}{4}$

Welfare is equal to $W = \frac{\theta}{2}\left(\sqrt{\frac{1}{4}} + \frac{1}{3}\frac{9}{4}\right) = \frac{5}{8}\theta$.

The risk-averse type will get $\frac{1}{4}$ with probability 1, and he will be indifferent to a lottery chosen by the risk-neutral type which gives $\frac{9}{4}$ with probability $\frac{1}{3}$ and zero otherwise.

c) If the planner could reveal types, he would solve the same problem, but no incentive-compatibility constraints. He won't need a lottery, but would just give each type a fixed amount.

$\sqrt{c_2} + c_3 \to \max \qquad \text{s.t.} \qquad c_2 + c_3 = 2e = 1$

FOC: $\frac{1}{2\sqrt{c_2}} = 1 \qquad c_2 = \frac{1}{4} \qquad c_3 = \frac{3}{4}$

Welfare: $W = \frac{\theta}{2}\sqrt{c_2} + \frac{\theta}{2}c_3 = \frac{\theta}{2}\frac{1}{2} + \frac{\theta}{2}\frac{3}{4} = \frac{5}{8}\theta$

So the planner achieved the first best allocation!