Economics 202B, TA section 1 Anton Cheremukhin

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1. **Overview of "Theory of Value" by G. Debreu.**

Goods/Choices to make. S: Commodity space

The dimension of the commodity space corresponds to the number of commodities, which are indexed by time, location and events (states of nature).

Producers $j = \overline{1, J}$.

Y_j: Production set y_j : firm j production decision.

The consumption set belongs to the commodity space. Outputs are ≥ 0. Inputs are ≤ 0.

Consumers $i = \overline{1, I}$.

X_i: Consumption set x_i : individual i consumption decision.

The consumption set belongs to the commodity space. Inputs are ≥ 0. Outputs are ≤ 0.

$\precsim_i$: Preferences

Preferences could be defined by any complete preordering. Usually definded using a utility function $u_i(x_i)$.

Additional features:

Endowment: $\Omega_i : \omega_i \geq 0, \quad \Sigma_i \omega_i = \omega \in S.$

Ownership: $\Theta_j : \theta_{ij} \geq 0, \quad \Sigma_i \theta_{ij} = 1$

Ownership is described by claims on profits of firm j owned by consumer i.

DEFINITIONS

An economy can be completely described by $E = \left\{ (X_i, \precsim_i, \omega_i)_{i=1}^{I}, (Y_j, \Theta_j)_{j=1}^{J} \right\}$.

Definition 1: (Feasiblity)

Allocation $(\{x_i\}, \{y_j\}) = (x, y)$ is feasible iff:

1.1) $\forall i : x_i \in X_i$,

1.2) $\forall j : y_j \in Y_j$,

1.3) $\Sigma_i x_i - \Sigma_j y_j = \Sigma_i \omega_i$.

Definition 2: (Pareto optimum)

Allocation (x, y) is a pareto-optimum (PO) iff:

2.1) (x, y) is feasible

2.2) $\nexists (\tilde{x}, \tilde{y})$:

- 2.2a) $(\tilde{x}, \tilde{y})$ is feasible
- 2.2b) $\forall i : \tilde{x}_i \succsim_i x_i \quad (u_i(\tilde{x}_i) \geq u_i(x_i))$
- 2.2c) $\exists i_0 : \tilde{x}_{i_0} \succ_i x_{i_0} \quad (u_i(\tilde{x}_i) > u_i(x_i))$

Definition 3: (Competitive equilibrium)

A tuple: $\left(x_i^*, y_j^*, p^*\right) \in \left(X_i, Y_j, R^{\dim S}\right)$ is a competitive equilibrium (CE) iff:

3.1) $x_i^* = \arg\max_{x_i \in X_i} \left\{ u(x_i) | \, p^* x_i \leq p^* \omega_i + \Sigma_j \theta_{ij} p^* y_j^* \right\}$

3.2) $y_j^* = \arg\max_{y_j \in Y_j} p^* y_j$

3.3) $\Sigma_i x_i^* - \Sigma_j y_j^* = \Sigma_i \omega$

RESULTS

Proposition 4: (Existence of equilibrium)

4.1) X_i closed, convex, bounded from below.

4.2a) $\nexists x_i' \in X_i$ s.t. $\forall x_i \in X_i : u_i(x_i) \leq u_i(x_i')$. (No satiation).

4.2b) $\forall x_i' \in X_i : \{x_i \in X_i : u_i(x_i) \leq u_i(x_i')\}$ and $\{x_i \in X_i : u_i(x_i) \geq u_i(x_i')\}$ are closed in X_i (continuity).

4.2c) $\forall x_i^1, x_i^2 \in X_i, \forall t \in (0,1) : u_i(x_i^2) > u_i(x_i^1) \Rightarrow u_i(tx_i^2 + (1-t)x_i^1) > u_i(x_i^1)$. (Quasi-concavity).

4.3) $\forall i : \exists x_i^0 \in X_i : x_i^0 << \omega_i$

4.4a) $\forall j : 0 \in Y_j$ (possibility of inaction)

4.4b) Y closed and convex

4.4c) $Y \cap (-Y) \subset \{0\}$ (irreversibility)

4.4d) $(-\Omega) \subset Y$ (free disposal)

Proposition 5: (Existence of optimum)

5.1) X_i closed, connected, bounded from below.

4.2b) $\forall x_i' \in X_i : \{x_i \in X_i : u_i(x_i) \leq u_i(x_i')\}$ and $\{x_i \in X_i : u_i(x_i) \geq u_i(x_i')\}$ are closed in X_i (continuity).

5.3) Y closed, convex, $Y \cap \Omega \subset \{0\}$ (no free lunch).

5.4) $\omega \in X - Y$ (no waste).

Theorem 6: (First Welfare Theorem)

6.1) (x^*, y^*, p^*) is a CE

6.2) X_i convex

6.3) $\forall x_i^1, x_i^2 \in X_i, \forall t \in (0,1) : u_i(x_i^2) > u_i(x_i^1) \Rightarrow u_i(tx_i^2 + (1-t)x_i^1) > u_i(x_i^1)$. (Quasi-concavity).

6.4) $\nexists x_i' \in X_i$ s.t. $\forall x_i \in X_i : u_i(x_i) \leq u_i(x_i')$. (No satiation).

Then (x^*, y^*) is a PO.

Theorem 7: (Second Welfare Theorem)

7.1) (x^*, y^*) is a PO

7.2a) X_i convex

7.2b) $\forall x_i' \in X_i : \{x_i \in X_i : u_i(x_i) \leq u_i(x_i')\}$ and $\{x_i \in X_i : u_i(x_i) \geq u_i(x_i')\}$ are closed in X_i (continuity).

7.2c) $\forall x_i^1, x_i^2 \in X_i, \forall t \in (0,1) : u_i(x_i^2) > u_i(x_i^1) \Rightarrow u_i(tx_i^2 + (1-t)x_i^1) > u_i(x_i^1)$. (Quasi-concavity).

7.2d) x_i^* is not a satiation point for some i.

7.3) Y convex.

Then $\exists p^* \neq 0 :$ (x^*, y^*, p^*) is a CE (with $\omega_i = x_i^* - \frac{y^*}{m}, \theta_{ij} = \frac{1}{m}$)

7.4*) $px_i^* \neq \min pX_i$ in equilibrium

7.5*) S is finite-dimensional or Y has an interior point

Remark: another (alternative) condition for a CE to be PO (First WT) is local non-satiation: $\forall x \in X \forall \varepsilon > 0 \exists y \in X : \|y - x\| \leq \varepsilon$ and $y \succ x$

(For any point x in the consumption set there is a preferred point y arbitrarily close).

Exercise 1 *An Economy with Lotteries*

There is a mass m of identical households with preferences $u(c,n) = \sqrt{c} - \sqrt{n}$ over consumption c and labor supply n. Labor supply can only take on two values $\{0,1\}$, corresponding to no employment and full-time employment. There a single firm in the economy that operates a technology $f(z,n) = zn$ which transforms labor n into the consumption good. Here z is a productivity shock which takes on values $\{0.5, 1\}$ with equal probability. Notice that z is an aggregate shock, since there is only one firm.

a) Choose a suitable commodity space for this economy. Define the consumption set, the production set and the resource constraint.

b) Are the conditions for Welfare Theorems satisfied?

c) Find the competitive equilibrium allocation and prices.

d) Could the consumer do any better if any $n \in [0,1]$ could be supplied?

Solution:

$S = R^4 = \left\{c_{z=\frac{1}{2}}, c_{z=1}, \pi^{emp}_{z=\frac{1}{2}}, \pi^{emp}_{z=1}\right\}$

$X = \{x \in S | x_1 \geq 0, x_2 \geq 0, -x_3 \in [0,1], -x_4 \in [0,1]\}$

$Y = \left\{y \in S | y_1 \leq -\frac{1}{2}y_3, y_2 \leq -y_4, y_3 \leq 0, y_4 \leq 0\right\}$

$u(x) = \frac{1}{2}\left(\sqrt{x_1} + x_3\right) + \frac{1}{2}\left(\sqrt{x_2} + x_4\right)$

$RC: \quad mx = y$

CE: $\quad (x^*, y^*, p^*) \quad s.t.$

1) $\quad x^* = \arg\max_{x\in X} \{u(x) | p^*x \leq 0\}$

2) $\quad y^* = \arg\max_{y\in Y} p^*y$

3) $\quad mx^* - y^* = 0$

WT1: X_i is convex, preferences are strictly increasing and quasi-concave.

WT2: X_i is convex, continuous, strictly increasing and quasi-concave. Y is convex.

Solve PO problem: $\quad \max_{x\in X} \{u(x) | mx = y\}$

$u(x) = \frac{1}{2}\left(\sqrt{c_1} - \pi_1\right) + \frac{1}{2}\left(\sqrt{c_2} - \pi_2\right) \quad s.t. \quad mc_1 = \frac{1}{2}m\pi_1, \quad mc_2 = m\pi_2$

$u(x) = \frac{1}{2}\left(\sqrt{c_1} - 2c_1\right) + \frac{1}{2}\left(\sqrt{c_2} - c_2\right)$

FOCs: $\quad \frac{1}{2\sqrt{c_1}} = 2 \quad \frac{1}{2\sqrt{c_2}} = 1$

Hence, $\quad c_1 = \frac{1}{16} \quad c_2 = \frac{1}{4} \quad \pi_1 = \frac{1}{8} \quad \pi_2 = \frac{1}{4}$

$\boxed{x = \left(\frac{1}{16}, \frac{1}{4}, -\frac{1}{8}, -\frac{1}{4}\right) \qquad y = \left(\frac{m}{16}, \frac{m}{4}, -\frac{m}{8}, -\frac{m}{4}\right)}$

Prices: $\quad \max_{x\in X} u(x) + \lambda px$

FOCs: $\quad u_i'(x) + \lambda p_i = 0 \qquad u'(x) = \left[\frac{1}{4\sqrt{x_1}}, \frac{1}{4\sqrt{x_2}}, \frac{1}{2}, \frac{1}{2}\right]$

Hence, $\quad \frac{p_3}{p_1} = \frac{u_3'(x)}{u_1'(x)} = \frac{4\sqrt{x_1}}{4\sqrt{x_2}} = \frac{1}{2} \qquad \frac{p_3}{p_1} = \frac{u_3'(x)}{u_1'(x)} = 2\sqrt{x_1} = \frac{1}{2} \qquad \frac{p_4}{p_1} = \frac{u_4'(x)}{u_1'(x)} = 2\sqrt{x_1} = \frac{1}{2}$

$\boxed{p = (2,1,1,1)}$

The result could not be improved upon if we allowed for $n \in [0,1]$.

Welfare (with lotteries): $\quad W = m\left(\frac{1}{2}\left(\sqrt{\frac{1}{16}} - \frac{1}{8}\right) + \frac{1}{2}\left(\sqrt{\frac{1}{4}} - \frac{1}{4}\right)\right) = \frac{3}{16}m$

Welfare (without): $\quad W = \max m\left(\frac{1}{2}\left(\sqrt{\frac{c_1}{2}} - \sqrt{c_1}\right) + \frac{1}{2}\left(\sqrt{c_2} - \sqrt{c_2}\right)\right) = 0$

Exercise 2 *Lotteries and indivisibilities in general equilibrium*

There is a measure one of ex ante identical people. A person can stay at home, work in the market sector, or be a manager. If a person manages n workers, the output of that person's production unit is n^{θ} where $0 < \theta < 1$. The utility function of an individual is $\log c - v(h)$ where $c > 0$ and $h \in \{0, 0.4, 0.7\}$ (home,market,manager). Use S=R^3={consumption, prob(worker), prob(manager)}. Specify X, Y, preferences, and conditions for FWT. What condition on $v()$ ensures $x_2 + x_3 < 1$ (some people don't work in equilibrium).

Solution:

(a) Commodity space $S = \{x_1, x_2, x_3\} = R^3$

Consumption set $X = \{x \in S|\, x_1 \geq 0, x_2 \leq 0, x_3 \leq 0, -x_2 - x_3 \leq 1\}$

$\precsim_i$: $U(x) = \log(x_1) - (1 + x_2 + x_3)\, v(0) + x_2 v(0.4) + x_3 v(0.7) =$

$\log(x_1) + x_2 (v(0.4) - v(0)) + x_3 (v(0.7) - v(0)) - v(0) = \log(x_1) - a(-x_2) - b(-x_3) - c$

(b) Fraction $-x_2$ are workers, fraction $-x_3$ are managers. Hence, each of the $-x_3$ managers produces $n^{\theta} = (x_2/x_3)^{\theta}$ output. The total output is $-x_3 (x_2/x_3)^{\theta} = (-x_2)^{\theta} (-x_3)^{1-\theta}$.

Production set $Y = \left\{ y \in S|\, y_2 \leq 0, y_3 \leq 0, y_1 \leq (-y_2)^{\theta} (-y_3)^{1-\theta} \right\}$

Resource constraint $X \leq Y$ Economy: $E = \{(X, \precsim), (Y)\}$

(c) Competitive equilibrium is a tuple: $(x^*, y^*, p^*) \in R^{3*3}$ such that

a) $x^* = \arg\max_{x \in X} \{U(x)|\, p^* x \leq p^* y^*\}$

b) $y^* = \arg\max_{y \in Y} p^* y$ c) $x^* = y^*$

Conditions for the first welfare theorem are: X convex, U - q.concave, no satiation.

By definition of X it is convex. Functions $\log(x)$, ax and bx are strictly increasing - no satiation of $U(.)$. A sum of a strictly concave function $\log(.)$ and a linear function is quasi-concave which accomplishes the proof.

Another way is to prove local non-satiation:

$\forall x = (x_1, x_2, x_3) \in X$ we can construct a sequence $x_n = \left(x_1 + \frac{1}{n}, x_2, x_3\right) \to x : x_n \succ x, \forall n.$

So for any point in X there is a preferred point x_n arbitrarily close (take big enough n).

(d) $\max_{x_i} \left\{ \log(x_1) - a x_2 - b x_3 - c| \quad x_1 \leq x_2^{\theta} x_3^{1-\theta}, \quad x_2 + x_3 \leq 1, \quad x_i \geq 0 \right\}$

$L = \log(x_1) - a x_2 - b x_3 - c + \lambda \left[x_2^{\theta} x_3^{1-\theta} - x_1 \right]$

FOC: $\lambda x_1 = 1$ $\quad a = \lambda x_2^{\theta} x_3^{1-\theta} \frac{\theta}{x_2}$ $\quad b = \lambda x_2^{\theta} x_3^{1-\theta} \frac{1-\theta}{x_3}$ $\quad$ RC: $x_1 = x_2^{\theta} x_3^{1-\theta}$

$x_2 = \frac{\theta}{a}, \quad x_3 = \frac{1-\theta}{b} \quad \Rightarrow x_2 + x_3 = \frac{\theta}{a} + \frac{1-\theta}{b} < 1 \Leftrightarrow \quad \boxed{\frac{\theta}{v(0.4) - v(0)} + \frac{1-\theta}{v(0.7) - v(0)} < 1}$