Econ202B Answer Key for Homework 4 — Anton Cheremukhin

March 14, 2008

Exercise 1 *Voting on public good provision*

$u(c,g) = \frac{c^{1-\sigma}}{1-\sigma} + \frac{g^{1-\sigma}}{1-\sigma} \qquad s.t. \qquad c = (1-\tau)\, y_i \qquad g = \tau\bar{y}$

Reduced-form policy preferences are:

$W(\tau, y^i) = \frac{((1-\tau)y_i)^{1-\sigma}}{1-\sigma} + \frac{(\tau\bar{y})^{1-\sigma}}{1-\sigma}$

$W'_\tau(\tau, y^i) = -((1-\tau)\, y_i)^{-\sigma} y_i + (\tau\bar{y})^{-\sigma}\bar{y} = 0 \qquad \Leftrightarrow \qquad \tau = 1/\left(1 + \left(\frac{\bar{y}}{y_i}\right)^{\frac{\sigma-1}{\sigma}}\right)$

$W''_\tau(\tau, y^i) = -\sigma((1-\tau)\, y_i)^{-\sigma-1} y_i^2 - \sigma(\tau\bar{y})^{-\sigma-1}\bar{y}^2 < 0$

W is continuous, concave and has a unique maximum.

Therefore, it is single-peaked. Hence the median voter is the Condorcet winner.

Each party chooses a policy in order to maximize the probability of winning.

Timing:

1) both parties simultaneously noncooperatively announce policies

2) voters choose between parties

3) the winner implements the announced policy

A **Majority Voting Equilibrium** is a Nash equilibrium of this game, when the probability of agent i voting for party A and B respectively is:

$$p_A^i = \begin{cases} 0 & W(\tau_A, y^i) < W(\tau_B, y^i) \\ \frac{1}{2} & W(\tau_A, y^i) = W(\tau_B, y^i) \\ 1 & W(\tau_A, y^i) > W(\tau_B, y^i) \end{cases} \qquad p_B = 1 - p_A$$

We shall use the **Proposition**: if preferences are single-peaked for a given ordering of policy alternatives, then a Condorcet winner always exists and coincides with the median voter.

Under majority voting the median voter would be the middle-income voter. Therefore,

$$\tau^* = 1/\left(1 + \left(\frac{y_M}{\bar{y}}\right)^{\frac{1-\sigma}{\sigma}}\right)$$

A **Probabilistic Voting Equilibrium** is a Nash equilibrium of this game, when the probability of agent i voting for party A and B respectively is:

$$p_A^i = F^i\left(W(\tau_A, y^i) - W(\tau_B, y^i)\right) \qquad p_B^i = 1 - p_A^i$$

The probability of party A winning is the average of the probabilities of all agents:

$\pi_A = \frac{1}{N}\Sigma_{i=1}^N F^i\left(W(\tau_A, y^i) - W(\tau_B, y^i)\right) \to \max_{\tau_A}$

FOC: $\quad \Sigma_{i=1}^N F^{i\prime}\left(W(\tau_A, y^i) - W(\tau_B, y^i)\right) W'_\tau(\tau_A, y^i) = 0$

Both parties will choose the same platform $\tau_A = \tau_B$, so $W(\tau_A, y^i) = W(\tau_B, y^i)$

FOC: $\quad 0 = \Sigma_{i=1}^N F^{i\prime}(0)\, W'_\tau(\tau_A, y^i) = -\Sigma_{i=1}^N F^{i\prime}(0)\left((1-\tau)\, y_i\right)^{-\sigma} y_i + \Sigma_{i=1}^N F^{i\prime}(0)(\tau\bar{y})^{-\sigma}\bar{y}$

$\left(\Sigma_{i=1}^N F^{i\prime}(0)(y_i)^{1-\sigma}\right)^{-\frac{1}{\sigma}} = \tau\left(\bar{y}^{-\frac{1-\sigma}{\sigma}}\left(\Sigma_{i=1}^N F^{i\prime}(0)\right)^{-\frac{1}{\sigma}} + \left(\Sigma_{i=1}^N F^{i\prime}(0)(y_i)^{1-\sigma}\right)^{-\frac{1}{\sigma}}\right)$

$$\tau^* = \frac{\left(\Sigma_{i=1}^N F^{i\prime}(0)(y_i)^{1-\sigma}\right)^{-\frac{1}{\sigma}}}{\left(\overline{y}^{-\frac{1-\sigma}{\sigma}}\left(\Sigma_{i=1}^N F^{i\prime}(0)\right)^{-\frac{1}{\sigma}}+\left(\Sigma_{i=1}^N F^{i\prime}(0)(y_i)^{1-\sigma}\right)^{-\frac{1}{\sigma}}\right)}$$

So agents will be weighted by their marginal densities $F^{i\prime}(0)$ in the equilibrium point. For instance, when all $F^{i\prime}(0)$ are the same, this is simplified to:

$$\tau^* = \overline{y^{1-\sigma}}^{-\frac{1}{\sigma}} / \left(\overline{y}^{-\frac{1-\sigma}{\sigma}} + \overline{y^{1-\sigma}}^{-\frac{1}{\sigma}}\right) = 1/\left(1+\left(\frac{\overline{y^{1-\sigma}}}{\overline{y}^{1-\sigma}}\right)^{\frac{1}{\sigma}}\right)$$

The answers coincide if $\overline{y^{1-\sigma}} = y_M$, or if $\overline{y} = y_M$ and $\sigma = 0$, or if $\sigma = 1$.

Exercise 2 *Dynamic Voting*

Bellman Equation:
$v_i(p,\Psi) = \frac{((1-\Psi(p))y_i)^{1-\sigma}}{1-\sigma} + \frac{(\Psi(p)[py_P+(1-p)y_R])^{1-\sigma}}{1-\sigma} + \beta\left[p' v_P(p',\Psi) + (1-p')\, v_R(p',\Psi)\right]$
s.t. $\quad p' = \frac{1}{2}\left(1+\Psi(p)^2\right)$
One-shot deviation:
$\tilde{v}_P(p,\tau,\Psi) = \frac{((1-\tau)y_P)^{1-\sigma}}{1-\sigma} + \frac{(\tau[py_P+(1-p)y_R])^{1-\sigma}}{1-\sigma} + \beta\left[p' v_P(p',\Psi) + (1-p')\, v_R(p',\Psi)\right]$
s.t. $\quad p' = \frac{1}{2}\left(1+\tau^2\right)$
Politico-economic equilibrium: $\{v_i(p,\Psi), \Psi(p)\}$ s.t.
1) $v_i(p,\Psi)$ is a solution of the Bellman equation (individual optimality) given Ψ
2) $\Psi(p) = \arg\max_\tau \tilde{v}_P(p,\tau,\Psi)$ (policy optimality) given v_i
Algorithm to solve the computational question on the homework:
1) set a grid of tax rates τ_i
2) set a grid of corresponding probabilities $p_i = \frac{1}{2}\left(1+\tau_i^2\right)$
3) set an initial guess $\Psi_0(p)$: outside loop until $\Psi_0(p) = \Psi_1(p)$
4) set initial guesses for value functions $v_{P0}(p), v_{R0}(p)$:
inside loop until $v_{P0}(p) = v_{P1}(p)$, $v_{R0}(p) = v_{R1}(p)$
5) compute $p'(\Psi_0(p))$
6) compute $v_{P1}(p)$ using y_P, $v_{P0}(p)$, $v_{R0}(p)$, $\Psi_0(p)$ and $p'(\Psi_0(p))$
7) compute $v_{R1}(p)$ using y_R, $v_{P0}(p)$, $v_{R0}(p)$, $\Psi_0(p)$ and $p'(\Psi_0(p))$
8) iterate inside loop untill convergence
9) take a grid of τ^i
10) take the results v_{P1} and v_{R1}
11) for each value on that grid τ^i compute $\tilde{v}(p,\tau^i)$
12) for each p find the argmaximum of this function $\tau(p)$
13) this is your new function $\Psi_1(p) = \tau(p)$
14) iterate over Ψ until convergence
It is useful to use a spline function:
spline(x,y,xx) approximates the matrix-correspondence y(x) at the point xx (xx is in between grid points of x).

The static case is the solution to a once-and-forever tax rate, that is consistent with the income distribution generated by that tax rate. The median voter is the poor voter, so the tax rate (of the median voter) in the static case is defined by:

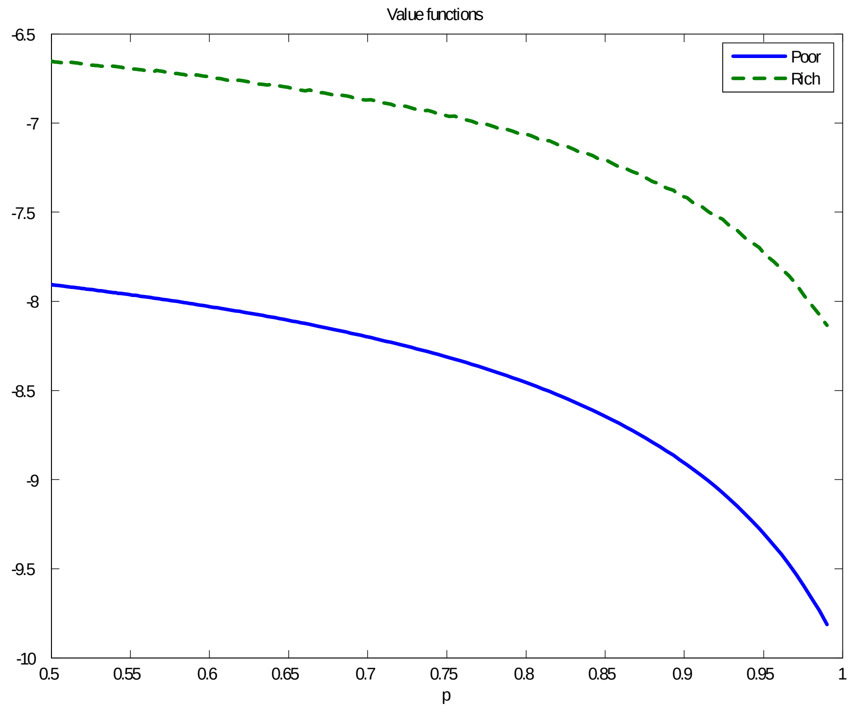
$$\tau = \frac{1}{1+\left(\frac{py_P+(1-p)y_R}{y_P}\right)^{\frac{\sigma-1}{\sigma}}}$$

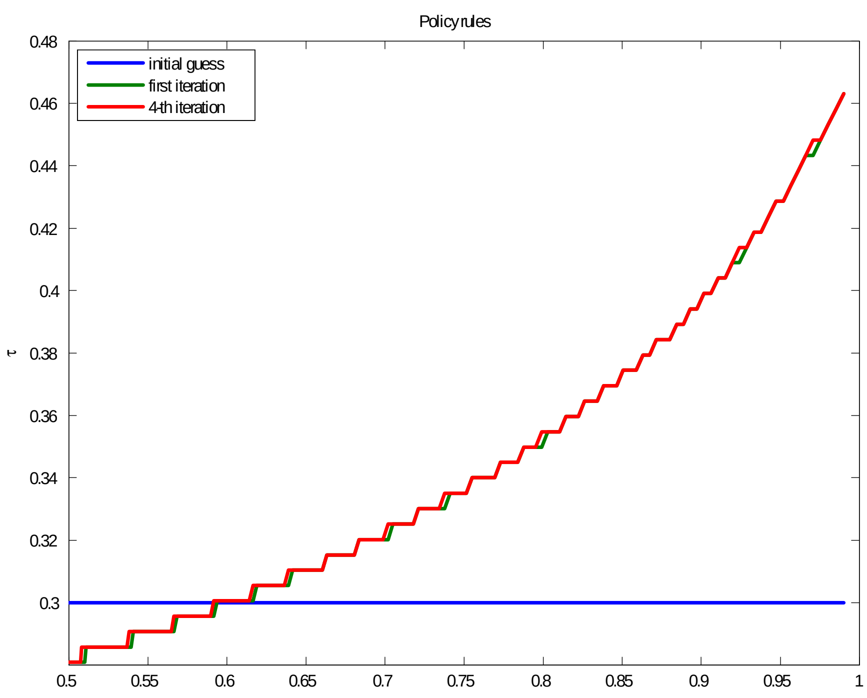
This tax rate has to be consistent with the income distribution defined by: $p = \frac{1}{2}(1+\tau^2)$

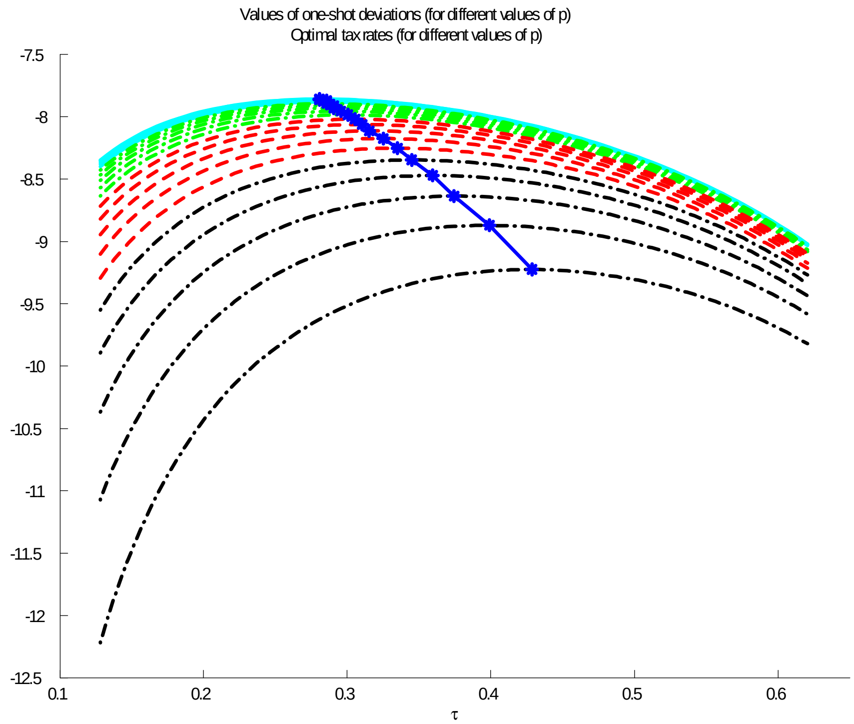
This system in our case boils down to: $p = \frac{1}{2}\left(1 + \left(\frac{1}{1+\sqrt{10-9p}}\right)^2\right)$

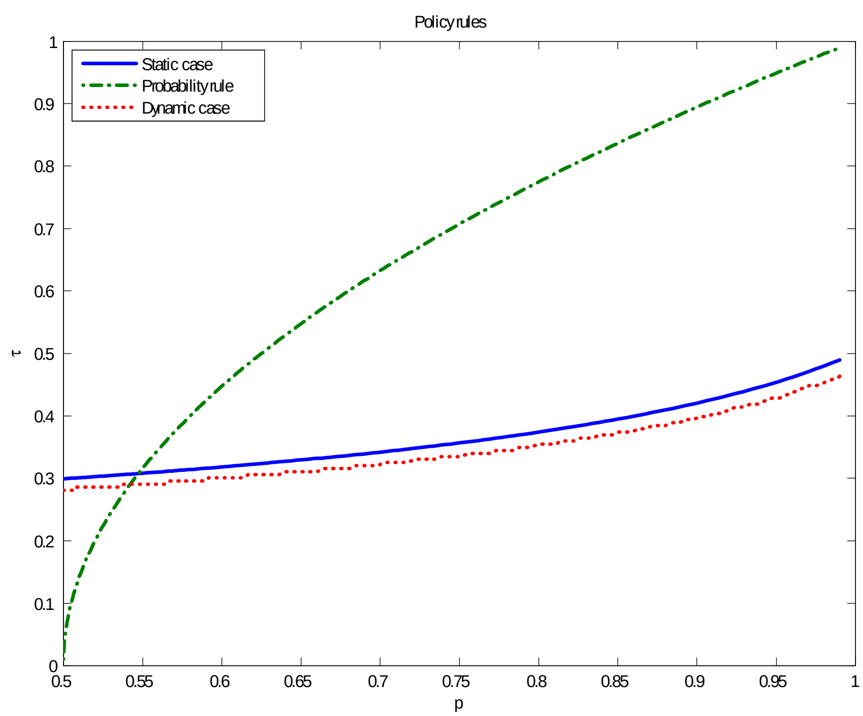
The only solution that fits the restriction on the probability is: $p = 0.55$. The corresponding tax rate is $\tau = 0.31$.

The steady-state tax rate in the dynamic setup is around $\tau = 0.29$. The steady-state fraction of the poor is $p = 0.54$.









So the chosen tax rate in the dynamic setup is lower than in the static case for any income distribution. This is because in the dynamic setting the poor voters (who essentially decide upon the tax) internalize the effect of the change in tax on the probability of becoming rich next period. In the static case nobody changes their type.

Exercise 3 *The AK Model*

Planner's problem:
$\Sigma_{t=0}^{\infty}\beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \rightarrow \max$ s.t. $c_t + k_{t+1} = (1-\delta) k_t + Ak_t$
Consumer's problem:
$\Sigma_{t=0}^{\infty}\beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \rightarrow \max$ s.t. $c_t + k_{t+1} = (1-\delta) k_t + r_t k_t$
Firm's problem: $Ak_t - r_t k_t \rightarrow \max_{k_t}$
Competitive equilibrium: allocation $\{c_t, k_t\}$ and price $\{1, r_t\}$ that are individually optimal for the consumer and the firm given prices and satisfy the resource constraints. (2 markets, 2 prices).
Bellman equation: $v(k) = \max_{k'} \left[\frac{((A+1-\delta)k-k')^{1-\sigma}}{1-\sigma} + \beta v(k')\right]$
FOC: $((A+1-\delta)k - k')^{-\sigma} = \beta v'(k')$
ENV: $v'(k) = ((A+1-\delta)k - k')^{-\sigma}(A+1-\delta)$
Guess the form of the solution: $v(k) = \alpha \frac{k^{1-\sigma}}{1-\sigma}$
Then, $v'(k) = \alpha k^{-\sigma} = ((A+1-\delta)k - k')^{-\sigma}(A+1-\delta)$
On a balanced growth path $k' = (1+\gamma)k$.
Hence, it must be the case that: $\alpha = (A-\delta-\gamma)^{-\sigma}(A+1-\delta)$
For the first-order condition to be satisfied in steady-state, it must be the case that:
$\alpha = \left(\frac{c}{k}\right)^{-\sigma}(A+1-\delta) = (A-\delta-\gamma)^{-\sigma}(A+1-\delta) \quad \Rightarrow \quad \frac{c}{k} = A-\delta-\gamma$
Also in a steady-state: $(1+\gamma)^{\sigma} = \beta(A+1-\delta). \quad \Rightarrow \quad \boxed{\gamma = [\beta(A+1-\delta)]^{\frac{1}{\sigma}} - 1}$
$\Rightarrow \quad \alpha = (A-\delta-\gamma)^{-\sigma}(A+1-\delta) = \left((A+1-\delta)^{1-\frac{1}{\sigma}} - \beta^{\frac{1}{\sigma}}\right)^{-\sigma}$

$$\boxed{v(k) = \frac{k^{1-\sigma}}{1-\sigma}\left((A+1-\delta)^{1-\frac{1}{\sigma}} - \beta^{\frac{1}{\sigma}}\right)^{-\sigma}}$$

Exercise 4 *Growth with human and physical capital.*

Planner's problem:
$\Sigma_{t=0}^{\infty}\beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \rightarrow \max_{\{c_t,k_{t+1},h_{t+1}\}}$ *s.t.*
$c_t + k_{t+1} - (1-\delta)k_t + h_{t+1} - (1-\delta)h_t \leq Ak_t^{\alpha}h_t^{1-\alpha}$
Consumer's problem:
$\Sigma_{t=0}^{\infty}\beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \rightarrow \max_{\{c_t,k_{t+1},h_{t+1}\}}$ *s.t.*
$c_t + k_{t+1} - (1-\delta)k_t + h_{t+1} - (1-\delta)h_t \leq w_t h_t + r_t k_t$
Firm's problem:
$Ak_t^{\alpha}h_t^{1-\alpha} - w_t h_t - r_t k_t \rightarrow \max_{h_t,k_t}$
Competitive equilibrium: allocation $\{c_t, k_t, h_t\}$ and price $\{1, r_t, w_t\}$ that is individually optimal for the consumer and the firm given prices and satisfy the resource constraints. (3 markets, 3 prices).
Bellman equation: $v(k,h) = \max_{k',h'}\left[\frac{c^{1-\sigma}}{1-\sigma} + \beta v(k',h')\right]$
s.t. $c = Ak^{\alpha}h^{1-\alpha} + (1-\delta)k - k' + (1-\delta)h - h'$
The value function is homogeneous of degree $1-\sigma$:
If we increase k, h, k', h' by a factor λ, we get $c\lambda$ instead of c.

$$v(\lambda k, \lambda h) = \max_{k',h'}\left[\lambda^{1-\sigma}\frac{c^{1-\sigma}}{1-\sigma} + \beta v(\lambda k', \lambda h')\right]$$

Also by multiplying the Bellman equation by $\lambda^{1-\sigma}$ we get that:

$$\lambda^{1-\sigma} v(k,h) = \max_{k',h'} \left[\lambda^{1-\sigma} \frac{c^{1-\sigma}}{1-\sigma} + \beta \lambda^{1-\sigma} v(k',h')\right]$$

Hence, $v(\lambda k, \lambda h) = \lambda^{1-\sigma} v(k,h)$ for any k, h.
Optimality conditions:
FOC: $c^{-\sigma} = \beta v_k'(k',h')$
FOC: $c^{-\sigma} = \beta v_h'(k',h')$
ENV: $v_k'(k,h) = c^{-\sigma}\left(\frac{\alpha}{k} A k^{\alpha} h^{1-\alpha} + 1 - \delta\right)$
ENV: $v_h'(k,h) = c^{-\sigma}\left(\frac{1-\alpha}{h} A k^{\alpha} h^{1-\alpha} + 1 - \delta\right)$
For a balanced growth path we need:
$\beta\left(\frac{\alpha}{k} A k^{\alpha} h^{1-\alpha} + 1 - \delta\right) = \beta\left(\frac{1-\alpha}{h} A k^{\alpha} h^{1-\alpha} + 1 - \delta\right) \quad \Rightarrow \quad \frac{\alpha}{k} = \frac{1-\alpha}{h}$
In steady-state: $\frac{h}{k} = \frac{1-\alpha}{\alpha} \quad \Rightarrow \quad \frac{c}{k} = A \frac{h}{k}^{1-\alpha} - \delta - \delta \frac{h}{k} = \frac{1}{\alpha}\left[A\alpha^{\alpha}(1-\alpha)^{1-\alpha} - \delta\right]$
The growth rate satisfies: $\beta\left(A\alpha^{\alpha}(1-\alpha)^{1-\alpha} + 1 - \delta\right) = (1+\gamma)^{\sigma} \quad \Rightarrow$

$$\boxed{\gamma = \left[\beta\left(A\alpha^{\alpha}(1-\alpha)^{1-\alpha} + 1 - \delta\right)\right]^{\frac{1}{\sigma}} - 1}$$

Transition dynamics:
FOC: $c^{-\sigma} = \beta(c')^{-\sigma}\left(\frac{\alpha}{k'} A (k')^{\alpha}(h')^{1-\alpha} + 1 - \delta\right)$
FOC: $c^{-\sigma} = \beta(c')^{-\sigma}\left(\frac{1-\alpha}{h'} A (k')^{\alpha}(h')^{1-\alpha} + 1 - \delta\right)$
Hence, $\boxed{\frac{\alpha}{k'} = \frac{1-\alpha}{h'}}$ at any point in time.
Use the expressions for consuption:
$c' = \left(A\left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha} + \frac{1-\delta}{\alpha}\right)k' - \frac{1}{\alpha}k'' \qquad c = Ak^{\alpha}h^{1-\alpha} + (1-\delta)k + (1-\delta)h - \frac{k'}{\alpha}$
Their ratio is constant at any point in time:
$\left(\frac{c'}{c}\right)^{\sigma} = \beta\left((1-\alpha)A\left(\frac{k'}{h'}\right)^{\alpha} + 1 - \delta\right) = \beta\left(A(\alpha)^{\alpha}(1-\alpha)^{1-\alpha} + 1 - \delta\right) = (1+\gamma)^{\sigma}$
Therefore, starting from the very first period, consumption will be growing at a constant rate:
$c' = \left(A\left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha} + \frac{1-\delta}{\alpha}\right)k' - \frac{1}{\alpha}k'' = c(1+\gamma) = \left(Ak^{\alpha}h^{1-\alpha} + (1-\delta)k + (1-\delta)h - \frac{k'}{\alpha}\right)(1+\gamma)$
From tomorrow on, $k'' = (1+\gamma)k'$. Hence,
$\left(A\left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha} + \frac{1-\delta}{\alpha}\right)k' - \frac{1}{\alpha}k'(1+\gamma) = \left(Ak^{\alpha}h^{1-\alpha} + (1-\delta)k + (1-\delta)h - \frac{k'}{\alpha}\right)(1+\gamma)$

$$\boxed{k' = \frac{Ak^{\alpha}h^{1-\alpha} + (1-\delta)k + (1-\delta)h}{A\left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha} + \frac{1-\delta}{\alpha}}(1+\gamma)}$$

Given any k_0 and h_0 we compute k' and h'. Afterwards, all variables will be growing at a constant rate. This defines the whole path. Investment is always divided proportionately. Capital evolves like in the AK model and it jumps right onto a balanced growth path.