Econ202B Answer Key for Homework 3 — Anton Cheremukhin

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Exercise 1 *Specific utility function (Chamley, 1986)*

Consider the nonstochastic model with capital and labor in this chapter and assume that the period utility function is given by: $u(c_t, l_t) = \frac{c_t^{1-\sigma}}{1-\sigma} + v(l_t)$, *where* $\sigma > 0$.

(a) Show that the optimal tax policy in this economy is to set capital taxes equal to zero in period 2 and from there on.

(b) Suppose there is uncertainty in the economy. Derive the optimal ex ante capital tax rate for $t > 1$.

a. We use the notations developed in the paragraph "constructing the Ramsey plan". We have

$$V(c,n,\Phi) \equiv u(c,1-n) + \Phi(u_c c - u_l n) \\ = c^{1-\sigma}\left(\frac{1}{1-\sigma} + \Phi\right) + v(1-n) - \Phi v_l n$$

This shows in particular that:

$$\frac{V_c(t+1)}{V_c(t)} = \frac{u_c(t+1)}{u_c(t)}$$

Now recall that the first of the difference equations defining the Ramsey plan is:

$$V_c(t) = \beta V_c(t+1)[F_k(t+1) + 1 - \delta] \quad t \geq 1$$

Also, the Ramsey plan is a competitive equilibrium, so that the Euler equation of the household problem is satisfied:

$$u_c(t) = \beta u_c(t+1)\left[(1-\tau_{t+1}^k)r_{t+1} + 1 - \delta\right] = \beta u_c(t+1)\left[(1-\tau_{t+1}^k)F_k(t+1) + 1 - \delta\right.$$

Combined with the previous equation and with the identity $\frac{V_c(t+1)}{V_c(t)} = \frac{u_c(t+1)}{u_c(t)}$, it implies that, for $t \geq 1$:

$$[F_k(t+1) + 1 - \delta] = \left[(1-\tau_{t+1}^k)F_k(t+1) + 1 - \delta\right]$$

So that $\tau_t^k = 0$ for all $t \geq 2$.

b. The reasoning is very similar. Under uncertainty, the first of the difference equations defining a Ramsey plan is:

$$V_c(s^t) = \beta \sum_{s_{t+1}} \pi(s_{t+1} \mid s^t) V_c(s_{t+1}, s^t)\left[F_k(s_{t+1}, s^t) + 1 - \delta\right] \quad t \geq 1$$

As before, the Ramsey plan is a competitive equilibrium, so that the Euler equation of the household problem is verified:

$$u_c(s^t) = \beta \sum_{s^{t+1}} \pi(s_{t+1} \mid s^t) u_c(s_{t+1}, s^t)\left[\left(1 - \tau^k(s_{t+1}, s^t)\right) r(s_{t+1}, s^t) + 1 - \delta\right] \\ = \beta \sum_{s^{t+1}} \pi(s_{t+1} \mid s^t) u_c(s_{t+1}, s^t)\left[\left(1 - \tau^k(s_{t+1}, s^t)\right) F_k(s_{t+1}, s^t) + 1 - \delta\right]$$

We use again the fact that:

$$\frac{V_c(s_{t+1}, s^t)}{V_c(s^t)} = \frac{u_c(s_{t+1}, s^t)}{u_c(s^t)}$$

And we find:

$$\beta \sum_{s_{t+1}} \pi(s_{t+1} \mid s^t) \frac{u_c(s_{t+1}, s^t)}{u_c(s^t)} \tau^k(s_{t+1}, s^t) F_k(s_{t+1}, s^t) = 0 \quad t \geq 1$$

With $p(s_{t+1} \mid s^t) = \beta\pi(s_{t+1} \mid s^t)\frac{u_c(s_{t+1},s^t)}{u_c(s^t)}$ and $r(s_{t+1}, s^t) = F_k(s_{t+1}, s^t)$, this can be rewritten:

$$\beta \sum_{s_{t+1}} p(s_{t+1} \mid s^t) \tau^k(s_{t+1}, s^t) = 0 \quad t \geq 1$$

i.e., the ex-ante capital tax is zero for $t \geq 2$.

Exercise 2 *Another specific utility function*

Consider the following optimal taxation problem. There is no uncertainty. There is one good that is produced by labor $1 - x_t$ of the representative household, and that can be divided among private consumption and government consuption. $c_t + g_t = 1 - x_t$. The good is produced by zero-profit competitive firms with a linear technology. A representative consumer maximizes $\Sigma_0^\infty \beta^t u(c_t, x_t)$ subject to the sequence of budget constraints $c_t + q_t b_{t+1} \leq (1 - \tau_t)(1 - x_t) + b_t$. Assume $u(c_t, x_t) = c_t - \frac{1}{2}(1 - x_t)^2$.

(a) Argue that in the competitive equilibrium $q_t = \beta$ and $x_t = \tau_t$.

(b) Assume zero initial bond holdings and derive the lifetime intertemporal constraint.

$$\Sigma_0^\infty \beta^t (c_t - (1 - x_t)(1 - \tau_t)) = 0.$$

Given an exogenous sequence of government purchases government maximizes welfare subject to its resource constraint $\Sigma_0^\infty \beta^t (g_t - (1 - x_t)\tau_t) = 0$ and the household's first-order condition $x_t = \tau_t$.

(c) Consider the process for $g_t = \begin{cases} 0 & t = 2k \\ \frac{1}{2} & t = 2k + 1 \end{cases}$. Solve for the Ramsey plan when $\beta = 0.95$. Show that the optimal tax rate is constant.

(d) Consider the process for $g_t = \begin{cases} \frac{1}{2} & t = 2k \\ 0 & t = 2k + 1 \end{cases}$. Solve for the Ramsey plan when $\beta = 0.95$. Show that the optimal tax rate is constant. Is it larger or lower than in the other case?

(e) Interpret your results in terms of "tax smoothing"?

(f) Under what conditions is the tax rate $\tau = 0$?

Solution:

(a) In a competitive equilibrium the consumer solves:

$\Sigma_0^\infty \beta^t \left(c_t - \frac{1}{2}(1-x_t)^2\right) \to \max_{c_t,x_t,b_{t+1}}$ s.t. $c_t + q_t b_{t+1} \leq (1-\tau_t)(1-x_t) + b_t$

The first-order conditions for optimality are:

$\lambda_t = u_{c_t} = 1 \qquad (1-x_t) = u_{x_t} = \lambda_t (1-\tau_t) \qquad \lambda_t q_t = \beta \lambda_{t+1}$

This implies, that in a competitive equilibrium $q_t = \beta$ and $x_t = \tau_t$.

(b) Take the sequence of budget constraints and multiply each by β^t and sum.

$\Sigma_0^\infty \beta^t [c_t - (1-\tau_t)(1-x_t) + q_t b_{t+1} - b_t] = 0$

$\Sigma_0^\infty \beta^t [q_t b_{t+1} - b_t] = \beta b_1 - b_0 + \beta[\beta b_2 - b_1] + \beta^2 [\beta b_3 - b_2] + ... = -b_0 = 0$

Assuming the initial bond holdings are equal to zero, this implies:

$\Sigma_0^\infty \beta^t [c_t - (1-\tau_t)(1-x_t)] = 0$

If we substitute the equilibrium condition $x_t = \tau_t$ we get: $\Sigma_0^\infty \beta^t \left[c_t - (1-x_t)^2\right] = 0$

The resource constraint is (g_t is given exogenously): $c_t + g_t = 1 - x_t$

(c-d) The Ramsey plan maximizes welfare subject to the implementability constraint and the resource constraint: $\Sigma_0^\infty \beta^t \left(c_t - \frac{1}{2}(1-x_t)^2\right) \to \max_{c_t,x_t,b_{t+1}}$

s.t. $\Sigma_0^\infty \beta^t \left[c_t - (1-x_t)^2\right] = 0$ s.t. $c_t + g_t = 1 - x_t$

$L = \Sigma_0^\infty \beta^t \left((1+\Phi) c_t - \left(\frac{1}{2} + \Phi\right)(1-x_t)^2 + \mu_t (1 - x_t - c_t - g_t)\right)$

FOC: $1 + \Phi = \mu_t \qquad (1+2\Phi)(1-x_t) = \mu_t \qquad \Rightarrow \qquad x_t = \frac{\Phi}{1+2\Phi}$.

The input of labor is constant, hence so is the optimal tax rate.

Consumption will reflect all the risk, because the worker is risk-neutral: $c_t = 1 - x - g_t$.

The implementability condition implies:

$\Sigma_0^\infty \beta^t c_t = \Sigma_0^\infty \beta^t (1-x_t)^2 = \frac{1}{1-\beta}(1-x)^2 \qquad \Sigma_0^\infty \beta^t c_t = \Sigma_0^\infty \beta^t (1-x-g_t) = \frac{1-x}{1-\beta} - \Sigma_0^\infty \beta^t g_t$

$(1-x)^2 - (1-x) + (1-\beta)\Sigma_0^\infty \beta^t g_t = 0$ Solution is: $1 - x = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4(1-\beta)\Sigma_0^\infty \beta^t g_t}$

$$\boxed{\tau = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4(1-\beta)\Sigma_0^\infty \beta^t g_t}}$$

The two solutions reflect the fact that the Laffer curve has a maximum lifetime revenue it can guarantee, and each revenue below that can be achieved using two different tax rates.

In one case (when you raise $\frac{1}{2}$ in period 1): $\Sigma_{t=0}^\infty \beta^t g_t = \beta \Sigma_{k=0}^\infty \beta^{2k} \frac{1}{2} = \frac{1}{2}\frac{\beta}{1-\beta^2} = \frac{190}{39}$

In this case $1 - 4(1-\beta)\Sigma_0^\infty \beta^t g_t = \frac{1}{39} > 0$ so the revenue stream can be raised using two values of the tax rate. $\tau = \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{1}{39}} = [0.42, 0.58]$

In the other case (when you raise $\frac{1}{2}$ in period 0) $\Sigma_{t=0}^\infty \beta^t g_t = \Sigma_{k=0}^\infty \beta^{2k} \frac{1}{2} = \frac{1}{2}\frac{1}{1-\beta^2} = \frac{200}{39}$

In this case $1 - 4(1-\beta)\Sigma_0^\infty \beta^t g_t = -\frac{1}{39} < 0$ so the revenue stream cannot be raised whatever the tax rate. The higher is the present value of consumption you want to raise, the higher the tax rate has to be (the one which is smaller). (e) The tax is constant over time, so there is tax-smoothing. All the risk will be reflected in consumption and bonds. $c_t = 1 - x - g_t \qquad b_{t+1} = \frac{1}{\beta}(g_t - \tau(1-\tau) + b_t)$

(f) For the optimal tax rate to be set to zero, the lifetime revenue to be raised should be zero. The path of g_t should be self-financing: $\Sigma_0^\infty \beta^t g_t = 0$.

Exercise 3 *Yet Another specific utility function*

$$g_0 = 0 \qquad g_1 = \begin{cases} 0 & \pi \\ G & 1-\pi \end{cases} \qquad b_0 = 0 \qquad c_i + g_i = n_i$$

$U = u(c_0) + v(1-n_0) + \beta E[u(c_1) + v(1-n_1)]$

(a) The consumer's problem is to maximize U given the constraints:

$c_0 + q_1 b_1 + q_G b_G = (1-\tau_0) n_0 \qquad c_1 = (1-\tau_1) n_1 + b_1 \qquad c_G = (1-\tau_G) n_G + b_G$

$\Rightarrow \qquad c_0 + q_1 (c_1 - (1-\tau_1) n_1) + q_G (c_G - (1-\tau_G) n_G) = (1-\tau_0) n_0$

Hence, $\quad u(c_0) + v(1-n_0) + \beta\pi[u(c_1) + v(1-n_1)] + \beta(1-\pi)[u(c_G) + v(1-n_G)] \to \max_{n_i, c_i}$

s.t. $\quad c_0 + q_1 (c_1 - (1-\tau_1) n_1) + q_G (c_G - (1-\tau_G) n_G) = (1-\tau_0) n_0$

FOC: $\quad u'(c_0) = \lambda \qquad \beta\pi u'(c_1) = q_1 \lambda \qquad \beta(1-\pi) u'(c_G) = q_G \lambda$

FOC: $\quad \lambda(1-\tau_0) = v'(1-n_0) \qquad$ FOC: $\quad \lambda q_1 (1-\tau_1) = \beta\pi v'(1-n_1)$

FOC: $\quad \lambda q_G (1-\tau_G) = \beta(1-\pi) v'(1-n_G) \qquad$ In other words: $\quad \boxed{u'(c_i)(1-\tau_i) = v'(1-n_i)}$.

(b) The implementability condition is then:

$u'(c_0)(c_0 - (1-\tau_0) n_0) + \beta\pi u'(c_1)(c_1 - (1-\tau_1) n_1) + \beta(1-\pi) u'(c_G)(c_G - (1-\tau_G) n_G) = 0$

$u'(c_0) c_0 + \beta\pi u'(c_1) c_1 + \beta(1-\pi) u'(c_G) c_G = v'(1-n_0) n_0 + \beta\pi v'(1-n_1) n_1 + \beta(1-\pi) v'(1-n_1) n_G$

(c) The Ramsey plan maximizes welfare subject to the implementability constraint and the three resource constraints:

$$L = \left[\begin{array}{c} [u(c_0) + \Phi u'(c_0) c_0] + \beta\pi [u(c_1) + \Phi u'(c_1) c_1] + \\ + \beta(1-\pi)[u(c_G) + \Phi u'(c_G) c_G] + [v(1-n_0) - \Phi v'(1-n_0) n_0] + \\ + \beta\pi [v(1-n_1) - \Phi v'(1-n_1) n_1] + \beta(1-\pi)[v(1-n_G) - \Phi v'(1-n_1) n_G] + \\ + \psi_0 (n_0 - c_0 - g_0) + \beta\pi\psi_1 (n_1 - c_1 - g_1) + \beta(1-\pi)\psi_G (n_G - c_G - g_G) \end{array} \right]$$

FOC: $\quad [u(c_0) + \Phi u'(c_0) c_0]'_{c_0} = \psi_0 = -[v(1-n_0) - \Phi v'(1-n_0) n_0]'_{n_0}$

FOC: $\quad [u(c_1) + \Phi u'(c_1) c_1]'_{c_1} = \psi_1 = -[v(1-n_1) - \Phi v'(1-n_1) n_1]'_{n_1}$

FOC: $\quad [u(c_G) + \Phi u'(c_G) c_G]'_{c_G} = \psi_G = -[v(1-n_G) - \Phi v'(1-n_1) n_G]'_{n_G}$

Notice, that:

$[u(c) + \Phi u'(c) c]'_c = (1+\Phi) u'(c) + \Phi u''(c) c$

$[v(1-n) - \Phi v'(1-n) n]'_n = -(1+\Phi) v'(1-n) + \Phi u''(c) c$

Hence, $\quad (1+\Phi) u'(c) + \Phi u''(c) c = \psi = (1+\Phi) v'(1-n) - \Phi v''(1-n) n$

The Ramsey problem solves four equations for $\{n_0, n_1, n_G, \Phi\}$:

$(1+\Phi)[u'(n_i - g_i) - v'(1-n_i)] + \Phi[u''(n_i - g_i)(n_i - g_i) + v''(1-n_i) n_i] = 0 \qquad i = \{0, 1, G\}$

$u'(n_0 - g_0)(n_0 - g_0) + \beta\pi u'(n_1 - g_1)(n_1 - g_1) + \beta(1-\pi) u'(n_G - g_G)(n_G - g_G) =$

$= v'(1-n_0) n_0 + \beta\pi v'(1-n_1) n_1 + \beta(1-\pi) v'(1-n_1) n_G$

(d) Use the first-order conditions for optimality:

$(1+\Phi) u'(n_i - g_i) \tau_i + \Phi[u''(n_i - g_i)(n_i - g_i) + v''(1-n_i) n_i] = 0$

Since $u'() > 0 \quad v'() > 0 \quad u''() < 0 \quad v''() < 0$, it has to be the case that when the implementability constraint is binding ($\Phi > 0$), then all τ_i are non-negative. That means, all the government revenues are positive. We can use that to find the signs of the bond holdings:

$q_1 b_1 + q_G b_G + \tau_0 n_0 = n_0 - c_0 = g_0 = 0$

$\tau_1 n_1 - b_1 = n_1 - c_1 = g_1 = 0$

$\tau_G n_G - b_G = n_G - c_1 = g_G = G$

From equation 2 it follows that $b_1 \geq 0$. From equation 1 it then follows that $b_G \leq 0$. It does not contradict equation 3. The total bond holdings in period 0 have to be negative.

(e) Hence, the planner will borrow from the consumer in good states 1 and 0 to finance expenditure in the bad state G.

Exercise 4 *Ramsey taxation in the AK model*

(a) Consumer: $\max \Sigma_0^\infty \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$

s.t. $\Sigma_0^\infty q_t \left(c_t + k_{t+1} - (1-\delta) k_t\right) \leq \Sigma_0^\infty q_t (1-\tau_t) r_t k_t$

The version with bonds implies a budget constraint:

$c_t + k_{t+1} - (1-\delta) k_t + \frac{b_{t+1}}{R_t} \leq (1-\tau_t) r_t k_t + b_t$

Firm: $\max A k_t - r_t k_t$

Government constraint: $\tau_t r_t k_t + \frac{b_{t+1}}{R_t} = g_t + b_t$

Resource constraint: $c_t + k_{t+1} - (1-\delta) k_t + g_t \leq A k_t$

Solution of the consumer's problem:

FOC: $\beta^t c_t^{-\sigma} = q_t$

FOC: $q_t = q_{t+1} \left((1-\tau_{t+1}) r_{t+1} k_{t+1} + 1 - \delta\right)$

FOC: $\frac{q_t}{R_{t+1}} = q_{t+1}$

Firm's solution implies: $r_t = A$.

The Euler equations: $c_t^{-\sigma} = \beta c_{t+1}^{-\sigma} \left((1-\tau_{t+1}) A + 1 - \delta\right)$

$(1-\tau_{t+1}) A + 1 - \delta = R_{t+1}$

(b) Implementability constraint follows from the lifetime budget constraint:

$\Sigma_0^\infty \beta^t c_t^{-\sigma} \left(c_t + k_{t+1} - (1-\delta) k_t - (1-\tau_t) A k_t\right) = 0$

($b_0 = 0$ from the setup of the model).

Using the Euler equation for capital we can rewrite the equation as:

$\Sigma_0^\infty \beta^t c_t^{1-\sigma} = \left((1-\tau_0) A + 1 - \delta\right) k_0 c_0^{-\sigma}$

(c) The Ramsey problem is to maximize lifetime utility subject to the implementability condition and the resource constraint.

$\max \Sigma_0^\infty \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$ *s.t.* $\Sigma_0^\infty \beta^t c_t^{1-\sigma} = \left((1-\tau_0) A + 1 - \delta\right) k_0 c_0^{-\sigma}$

s.t. $c_t + k_{t+1} - (1-\delta) k_t + g_t \leq A k_t$

$L = \Sigma_0^\infty \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} + \Phi \frac{c_t^{1-\sigma}}{1-\sigma} + \mu_t \left[(A + 1 - \delta) k_t - c_t - k_{t+1} - g_t\right]\right] - \frac{\Phi}{1-\sigma} \left((1-\tau_0) A + 1 - \delta\right) k_0 c_0^{-\sigma}$

FOC: $c_t^{-\sigma} (1+\Phi) = \mu_t$ FOC$_0$: $c_0^{-\sigma} (1+\Phi) = \mu_0 - \sigma \frac{\Phi}{1-\sigma} \left((1-\tau_0) A + 1 - \delta\right) k_0 c_0^{-\sigma-1}$

FOC: $\mu_t = \beta \mu_{t+1} (A + 1 - \delta)$

(d) The solution has to satisfy: $c_t^{-\sigma} = \beta c_{t+1}^{-\sigma} (A + 1 - \delta)$.

At the same time the competitive equilibrium has to satisfy:

$c_t^{-\sigma} = \beta c_{t+1}^{-\sigma} \left((1-\tau_{t+1}) A + 1 - \delta\right)$

The two equations coincide only if $\tau_{t+1} = 0$ for any period $t > 0$.

That would also be true in a steady-state.

The only way to finance the government expenditure in this economy is to tax initial capital.

(e) If no bonds can be issued, then the government constraint is: $\tau_t r_t k_t = g_t$.

Given that the interest rate $r_t = A$, the tax rate is a function of g_t : $\tau_t = g_t / A k_t$.

In this case there is nothing to choose. The whole path is completely predetermined:

$c_t^{-\sigma} = \beta c_{t+1}^{-\sigma} \left(\left(1 - \frac{g_{t+1}}{A k_{t+1}}\right) A + 1 - \delta\right)$ $k_{t+1} = (A + 1 - \delta) k_t - g_t - c_t$

Hence, $(A + 1 - \delta) k_t - g_t - k_{t+1} = c_t$.

Therefore, the path of capital stocks has to satisfy:

$\left(\frac{(A+1-\delta) k_{t+1} - g_{t+1} - k_{t+2}}{(A+1-\delta) k_t - g_t - k_{t+1}}\right)^\sigma = \beta \left(\left(1 - \frac{g_{t+1}}{A k_{t+1}}\right) A + 1 - \delta\right)$

The path of $\{g_t\}$ completely determines the path of $\{k_t\}$ given k_0.