Econ202B Homework #2 Answer Key Anton Cheremukhin

February 11, 2008

Exercise 1 *Thomas and Worral meet Markov*

A household orders sequences $\{c_t\}_{t=0}^{\infty}$ of a single nondurable good by

$$E\sum_{t=0}^{\infty}\beta^t u(c_t), \beta \in (0,1)$$

where u is strictly increasing, twice continuously differentiable, and strictly concave with $u'(0) = +\infty$. The household receives an endowment of the consumption good of y_t that obeys a discrete state Markov chain with $P_{ij} = \text{Prob}(y_{t+1} = \overline{y}_j | y_t = \overline{y}_i)$, where the endowment y_t can take one of the I values $[\overline{y}_1, \dots, \overline{y}_I]$.

a. Conditional on having observed the time t value of the household's endowment, a social insurer wants to deliver expected discounted utility v to the household in the least cost way. The insurer observes y_t at the beginning of every period, and contingent on the observed history of those endowments, can make a transfer τ_t to the household. The transfer can be positive or negative and can be enforced without cost. Let $C(v, i)$ be the minimum expected discounted cost to the insurance agency of delivering promised discounted utility v when the household has just received endowment $\overline{y}_i$. (Let the insurer discount with factor β.) Write a Bellman equation for $C(v, i)$.

b. Characterize the consumption plan and the transfer plan that attains $C(v, i)$; find an associated law of motion for promised discounted value.

c. Now assume that the household is isolated and has no access to insurance. Let $v^a(i)$ be the expected discounted value of utility for a household in autarky, conditional on current income being $\overline{y}_i$. Formulate Bellman equations for $v^a(i), i = 1, \dots, I$.

d. Now return to the problem of the insurer mentioned in part b, but assume that the insurer cannot enforce transfers because each period the consumer is free to walk away from the insurer and live in autarky thereafter. The insurer must structure a history-dependent transfer scheme that prevents the household from every exercising the option to revert to autarky. Again, let $C(v, i)$ be the minimum cost for an insurer that wants to deliver promised value discounted utility v to a household with current endowment i. Formulate Bellman equations for $C(v, i), i = 1, \dots, I$. Briefly discuss the form of the law of motion for v associated with the minimum cost insurance scheme.

a. Let v_j be the promised value in state i. The Bellman equation for the planner is

$$C(v_i) = \min_{\tau, \{v'_j\}_{j=1}^{I}} \left\{ \tau_i + \beta \sum_j P_{ij} C(v'_j) \right\}$$

subject to the promise keeping constraint

$$v_i \leq u(y_i + \tau_i) + \beta \sum_j P_{ij} v'_j$$

b. The first order conditions are: w.r.t. $\tau_i : \lambda^{-1} = u_c(y_i + \tau_i)$ w.r.t. $v_j : \beta P_{ij} C'(v'_j) = \lambda \beta P_{ij} \quad \forall j \in \{1, ..., I\}$

The Benveniste-Scheinkman condition is $C'(v_i) = \lambda$. Combining equations we see that $C'(v'_j) = C'(v_i)$. Using the convexity of the cost function, $v'_j = v_i \quad \forall j \in \{1, ..., I\}$. So, for any given realization of today's income y_i, the planner offers a constant utiltiy promise in every state of nature tomorrow. From the first order condition we obtain the optimal transfer $\tau_i = u_c^{-1}\left(\frac{1}{C'(v_i)}\right) - y_i$.

c. The Bellman equation in autarky is:

$$v_i^a = u(y_i) + \beta \sum_j P_{ij} v_j^a$$

d. The problem with limited commitment imposes an additional participation constraint on the allocations. The Bellman equation for the planner is

$$C(v_i) = \min_{\tau, \{v'_j\}_{j=1}^{I}} \left\{ \tau_i + \beta \sum_j P_{ij} C(v'_j) \right\}$$

subject to the promise keeping constraint

$$v_i \leq u(y_i + \tau_i) + \beta \sum_j P_{ij} v'_j$$

and subject to the enforcement constraint

$$u(y_i + \tau_i) + \beta \sum_j P_{ij} v'_j \geq u(y_i) + \beta \sum_j P_{ij} v_j^a = v_i^a$$

The first order conditions now become:

w.r.t. $\tau_i : 1 = (\lambda + \eta) u_c(y_i + \tau_i)$ w.r.t. $v_j : \beta P_{ij} C'(v_j') = (\lambda + \eta)\beta P_{ij} \quad \forall j \in \{1, ..., I\}$ The Benveniste-Scheinkman condition is $C'(v_i) = \lambda$.

Therefore, $C'(v_j') \geq C'(v_i)$. In states of the world i where the enforcement constraint does not bind, a constant utility promise is optimal in every state of the world j: $v_j' = v_i \quad \forall j \in \{1, ..., I\}$. If the constraint binds in state i the planner decreases tomorrow's utility promise $v_j' < v_i$, but increases todays transfer $\tau_i = u_c^{-1}\left(\frac{1}{C'(v_j)}\right) - y_i$ compared to states where the constraint does not bind. The planner keeps the agent who threatens to walk away in the contract by increasing today's transfers while decreasing tomorrow's promises.

Exercise 2 *Private Information*

$u_1(c) = \begin{cases} c & c > 1 \\ \ln c + 1 & 0 < c \leq 1 \end{cases} \qquad u_2(c) = \begin{cases} c & c > 1 \\ 2c - 1 & 0 \leq c \leq 1 \end{cases}$

First best:

$u_1(\tau_1) + u_1(2 - \tau_1) + \beta u_2(1 - \tau_2) + \beta u_2(1 + \tau_2) \rightarrow \max$

$\frac{1}{\tau_1} = u_1'(\tau_1) = u_1'(2 - \tau_1) = 1 \qquad \tau_1 = 1 \qquad u_2'(1 - \tau_2) = u_2'(1 + \tau_2) \qquad \tau_2 = 0$

$\boxed{W_{FB} = \frac{1}{2}[1 + 1 + \beta + \beta] = 1 + \beta}$

(a) Constrained optimization

$\max_{\tau_1, \tau_2} \frac{1}{2}[u_1(\tau_1) + u_1(2 - \tau_1) + \beta u_2(1 - \tau_2) + \beta u_2(1 + \tau_2)]$

s.t. $\quad 0 < \tau_1 < 2, \qquad -1 \leq \tau_2 \leq 1$

s.t. $\quad c_1 + c_2 = \tau_1 + 2 - \tau_1 \leq 2 \qquad (\text{RC}_1)$ is satisfied automatically

s.t. $\quad c_3 + c_4 = 1 - \tau_2 + 1 + \tau_2 \leq 2 \qquad (\text{RC}_2)$ is satisfied automatically

s.t. $\quad u_1(2 - \tau_1) + \beta u_2(1 + \tau_2) \geq u_1(2 + \tau_1) + \beta u_2(1 - \tau_2) \qquad (\text{IC}_1)$

s.t. $\quad u_1(\tau_1) + \beta u_2(1 - \tau_2) \geq u_1(-\tau_1) + \beta u_2(1 + \tau_2) \qquad (\text{IC}_2)$ (holds because of the log)

Solution:

$L = u_1(\tau_1) + u_1(2 - \tau_1) + \beta u_2(1 - \tau_2) + \beta u_2(1 + \tau_2) +$
$+\lambda[u_1(2 - \tau_1) + \beta u_2(1 + \tau_2) - u_1(2 + \tau_1) - \beta u_2(1 - \tau_2)]$

FOC: $\quad \frac{\partial L}{\partial \tau_1} = u_1'(\tau_1) - u_1'(2 - \tau_1) - \lambda u_1'(2 - \tau_1) - \lambda u_1'(2 + \tau_1) = 0$

FOC: $\quad \frac{\partial L}{\partial \tau_2} = -\beta u_2'(1 - \tau_2) + \beta u_2'(1 + \tau_2) + \lambda[\beta u_2'(1 + \tau_2) + \beta u_2'(1 - \tau_2)] = 0$

Therefore, $u_1'(\tau_1) = (1 + \lambda) u_1'(2 - \tau_1) + \lambda u_1'(2 + \tau_1)$
$(1 - \lambda) u_2'(1 - \tau_2) = (1 + \lambda) u_2'(1 + \tau_2)$

Case 1: constraint is binding $\quad \lambda > 0, \tau_2 > 0$

Then, $u_2'(1 + \tau_2) = 1 \qquad u_2'(1 - \tau_2) = 2 \qquad (1 - \lambda) 2 = (1 + \lambda) 1$, Solution is: $\lambda = \frac{1}{3}$
$u_1'(\tau_1) = \left(1 + \frac{1}{3}\right) u_1'(2 - \tau_1) + \frac{1}{3} u_1'(2 + \tau_1)$

Case 1.1 $\quad \tau_1 < 1$

Then $u_1'(\tau_1) = \frac{1}{\tau_1} \qquad u_1'(2 - \tau_1) = u_1'(2 + \tau_1) = 1$

$\frac{1}{\tau_1} = \left(1 + \frac{1}{3}\right) + \frac{1}{3}$, Solution is: $\boxed{\tau_1 = \frac{3}{5}}$

$2 - \tau_1 + \beta(1 + \tau_2) - (2 + \tau_1) - \beta(2(1 - \tau_2) - 1) = 0$, Solution is: $\boxed{\tau_2 = \frac{2}{3\beta}\tau_1 = \frac{2}{5\beta}}$

$\boxed{W^{SB}_{\beta > \frac{2}{5}}} = \frac{1}{2} \ln(\tau_1) + 1 + 2 - \tau_1 + \beta(2(1 - \tau_2) - 1) + \beta(1 + \tau_2)\Big|_{\tau_1 = \frac{3}{5}, \tau_2 = \frac{2}{5\beta}} = \boxed{\beta + 1 - \frac{1}{2}\ln\frac{5}{3}}$

When $\frac{2}{5\beta} \geq 1$ i.e. $\boxed{\beta \leq \frac{2}{5}}$ the constraint $\tau_2 \leq 1$ is violated.

Then $\boxed{\tau_2 = 1}$ $\quad \boxed{\tau_1 = \frac{3}{2}\beta}$ since the IC_1 should hold as equality:

$(2 - \tau_1) + \beta(1 + 1) - (2 + \tau_1) - \beta(2(1 - 1) - 1) = 3\beta - 2\tau_1 = 0$

$\boxed{W^{SB}_{\beta \leq \frac{2}{5}}} = \frac{1}{2}\ln(\tau_1) + 1 + 2 - \tau_1 + \beta(2(1 - 1) - 1) + \beta(1 + 1)\big|_{\tau_1 = \frac{3}{2}\beta} = \boxed{\frac{1}{2}\ln\frac{3}{2}\beta - \frac{1}{4}\beta + \frac{3}{2}}$

Case 1.2 $\quad \tau_1 \geq 1$

Then $u_1'(\tau_1) = 1 \quad u_1'(2 - \tau_1) = \frac{1}{2 - \tau_1} \quad u_1'(2 + \tau_1) = 1$

$1 = \left(1 + \frac{1}{3}\right)\frac{1}{2 - \tau_1} + \frac{1}{3}$, Solution is: $\tau_1 = 0$ $\quad$ Contradiction.

Case 2 $\quad \lambda = 0$

$u_1'(\tau_1) = u_1'(2 - \tau_1) \quad u_2'(1 - \tau_2) = u_2'(1 + \tau_2) \quad \tau_2 = 0 \quad \tau_1 = 1$

$(2 - 1) + \beta(1) - (2 + 0) - \beta(2(1 - 0) - 1) = -1 < 0$ $\quad$ Contradiction.

(b) If there is a financial market, let the agents be able to borrow at the rate $R = \frac{1}{\beta}$.

Present-value budget constraints:

If type 1 tells the truth: $\quad c_1 = \tau_1 + b_1 \quad c_3 = 1 - \tau_2 - b_1\frac{1}{\beta} \Rightarrow \quad c_1 + \beta c_3 = \tau_1 + \beta - \beta\tau_2$

If type 1 lies: $\quad c_1 = -\tau_1 + b_1' \quad c_3 = 1 + \tau_2 - b_1'\frac{1}{\beta} \quad \Rightarrow \quad c_1 + \beta c_3 = -\tau_1 + \beta + \beta\tau_2$

For type 1 to tell the truth it is necessary, that $\tau_1 \geq \beta\tau_2$

If type 2 tells the truth: $\quad c_2 = 2 - \tau_1 + b_2 \quad c_4 = 1 + \tau_2 - b_2\frac{1}{\beta} \Rightarrow c_2 + \beta c_4 = 2 - \tau_1 + \beta + \beta\tau_2$

If type 2 lies: $\quad c_2 = 2 + \tau_1 + b_2' \quad c_4 = 1 - \tau_2 - b_2'\frac{1}{\beta} \quad \Rightarrow \quad c_2 + \beta c_4 = 2 + \tau_1 + \beta - \beta\tau_2$

For type 2 to tell the truth it is necessary, that $\tau_1 \leq \beta\tau_2$

Therefore, for the mechanism to be truth-tellin, it must be the case that: $\boxed{\tau_1 = \beta\tau_2}$.

That implies, that $c_1 + \beta c_3 = \beta \quad c_2 + \beta c_4 = 2 + \beta$ and the decisions do not depend on the transfer system.

Even in the absense of transfers: $\quad u_1(\beta - \beta c) + \beta u_2(c) \to \max_c$

$u_1'(\beta - \beta c) = u_2'(c) \quad c < 1 \quad \frac{1}{\beta - \beta c} = 2 \quad \Rightarrow \quad c = 1 - \frac{1}{2\beta} \quad \beta - \beta c = \frac{1}{2}$

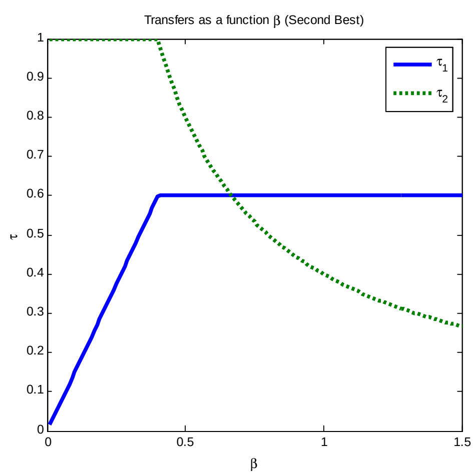
$W_1 = 1 - \log(2) + \beta\left(2\left(1 - \frac{1}{2\beta}\right) - 1\right) = \beta - \ln 2$

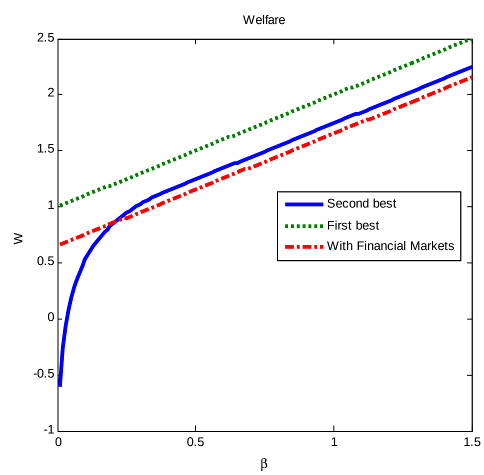
$u_1(2 + \beta - \beta c) + \beta u_2(c) \to \max_c \quad u_1'(2 + \beta - \beta c) = u_2'(c) \quad c > 1 \quad 1 = 1$

$W_2 = (2 + \beta - \beta c) + \beta(c) = \beta + 2 \quad \boxed{W^{FM}} = \frac{W_1 + W_2}{2} = \frac{2\beta + 2 - \ln 2}{2} = \boxed{\beta + 1 - \frac{1}{2}\ln 2}$

So the transfer system cannot improve upon the situation when financial markets are present.

Moreover, it is worse then the first best, and for high enough values of β it is worse then second best.





Exercise 3 *Thomas-Worrall meet Phelan-Townsend*

Consider the Thomas Worrall environment and denote $\Pi(y)$ the density of the i.i.d. endowment process, where y belongs to the discrete set of endowment levels $Y = [\overline{y}_1, \dots, \overline{y}_S]$. The one-period utility function is $u(c) = (1-\gamma)^{-1}(c-a)^{1-\gamma}$ where $\gamma > 1$ and $\overline{y}_S > a > 0$.

Discretize the set of transfers B and the set of continuation values W. We assume that the discrete set $B \subset (a - \overline{y}_S, \overline{b}]$. Notice that with the one period utility function above, the planner could never extract more than $a - \overline{y}_S$ from the agent. Denote $\Pi^v(b, w|y)$ the joint density over (b, w) that the planner offers the agent who reports y and to whom he has offered beginning of period promised value v. For each $y \in Y$ and each $v \in W$, the planner chooses a set of conditional probabililites $\Pi^v(b, w|y)$ to satisfy the Bellman equation

$$P(v) = \max_{\Pi^v(b,w,y)} \sum_{B \times W \times Y} [-b + \beta P(w)]\, \Pi^v(b, w, y) \tag{131}$$

subject to the following constraints:

$$\begin{aligned}
v &= \sum_{B\times W\times Y} [u(y+b) + \beta w]\, \Pi^v(b, w, y) \\
\sum_{B\times W} [u(y+b) + \beta w]\, \Pi^v(b, w|y) &\geq \sum_{B\times W} [u(y+b) + \beta w]\, \Pi^v(b, w|\tilde{y}) \\
& \quad \forall (y, \tilde{y}) \in Y \times Y \\
\Pi^v(b, w, y) &= \Pi(y)\Pi^v(b, w|y) \quad \forall (b, w, y) \in B \times W \times Y \\
\sum_{B\times W\times Y} \Pi^v(b, w, y) &= 1.
\end{aligned}$$

Here (132) is the promise keeping constraint, (132) are the truth-telling constraints, and (132), (132) are restrictions imposed by the laws of probability.

a. Verify that that given $P(w)$, one step on the Bellman equation is a linear programming problem.

b. Set $\beta = .94, a = 5, \gamma = 3$. Let S, N_B, N_W be the number of points in the grids for Y, B, W, respectively. Set $S = 10$, $N_B = N_W = 25$. Set $Y = \begin{bmatrix} 6 & 7 & \ldots 15 \end{bmatrix}$, $\text{Prob}(y_t = \overline{y}_s) = S^{-1}$. Set $W = [w_{\min}, \ldots, w_{\max}]$ and $B = [b_{\min}, \ldots, b_{\max}]$, where the intermediate points in W and B, respectively, are equally spaced. Please set $w_{\min} = \frac{1}{1-\beta}\frac{1}{1-\gamma}(y_{\min} - a)^{1-\gamma}$ and $w_{\max} = w_{\min}/20$ (these are negative numbers, so $w_{\min} < w_{\max}$). Also set $b_{\min} = (1 - y_{\max} + .33)$ and $b_{\max} = y_{\max} - y_{\min}$. For these parameter values, compute the optimal contract by formulating a linear program for one step on the Bellman equation, then iterating to convergence on it.

c. Notice the following probability laws:

$$\begin{aligned} \text{Prob}(b_t, w_{t+1}, y_t | w_t) &\equiv \Pi^{w_t}(b_t, w_{t+1}, y_t) \\ \text{Prob}(w_{t+1}|w_t) &= \textstyle\sum_{b \in B, y \in Y} \Pi^{w_t}(b, w_{t+1}, y) \\ \text{Prob}(b_t, y_t|w_t) &= \textstyle\sum_{w_{t+1} \in W} \Pi^{w_t}(b_t, w_{t+1}, y_t). \end{aligned} \tag{133}$$

Please use these and other probability laws to compute $\text{Prob}(w_{t+1}|w_t)$. Show how to compute $\text{Prob}(c_t)$, assuming a given initial promised value w_0. **d.** Assume that $w_0 \approx -2$. Compute and plot $F_t(c) = \text{Prob}(c_t \leq c)$ for $t = 1, 5, 10, 100$. Qualitatively, how do these distributions compare with those for the simple village and money lender model with no information problem and one-sided lack of commitment?

The problem is linear in constraints and the intra-period objective function. So one should use linear programming inside the period and iterate over different values of v to compute $P(v)$. The best way to do it is to use the same grid for v's as for ω's.

$P(v) = \max_{\Pi^v(b,\omega|y)} \Sigma_{b,\omega,y} \left[-\tau + \beta P(\omega)\right] \Pi^v(b,\omega|y)\, \Pi(y)$

s.t. $\quad v = \Sigma_{b,\omega,y} \left[u(c+b) + \beta\omega\right] \Pi^v(b,\omega|y)\, \Pi(y)$

s.t. $\quad \Sigma_{b,\omega} \left[u(c+b) + \beta\omega\right] \Pi^v(b,\omega|y) \geq \Sigma_{b,\omega} \left[u(c+b) + \beta\omega\right] \Pi^v(b,\omega|\tilde{y}) \qquad \forall y \forall \tilde{y}$

s.t. $\quad \Sigma_{b,\omega} \Pi^v(b,\omega|y) = 1 \qquad \forall y$

Notice, that:

a) the conditional probabilities should sum up to one for each income realization, not unconditionally

b) not only (though almost only) the downward constraints are binding in the numerical application

c) the grid for b given in the book implies negative consumption values, and given $\gamma = 3$, $1 - \gamma = -2$ is computable for any values of consumption, however, the result should not change if you set $\gamma = 2.99$. This means you should always check that consumption is nonnegative when computing the utility function, then you don't need transfers up to $b_{\min} = \{a - y_{\max} + 1.33\}$.

d) the interesting part of the values for promised utilities corresponds to high levels, so making the grid for ω equally spaced is not that good of an idea

e) having a 25x25x10 grid takes a LOT of time, so having it reduced to 10 or 15 can help a lot.

The **algorithm** is as follows:

1) set some initial guess for $P_0(\omega)$ say some linear decreasing funciton.

2) for a given $P_0(\omega)$ solve the maximization problem for each $v \in [\omega_1, ..., \omega_{N_\omega}]$ and obtain a vector of solutions $\Pi^v(b, \omega|y)$.

3) multiply the solution by the objective function to find $[P(\omega_1), ..., P(\omega_{N_\omega})]$ values for the $P_1(\omega)$ to be used at the next step.

4) Iterate over 1-3 until convergence, i.e. until $\|P_n(\omega) - P_{n-1}(\omega)\| \leq \varepsilon$

5) Plot the function $P_n(\omega)$

6) Use the last step solution $\Pi^v(b, \omega|y)$ to compute the transition matrix for ω :

$\Pr(\omega_t|\omega_{t-1}) = \Sigma_{b,y}\Pi^{\omega_{t-1}}(b, \omega_t|y)\Pi(y)$

7) Compute the distribution of consumption in period t given the initial promised value ω_0

$\Pr(c_t) = \Pi(b_t, y_t|b_t + y_t \leq c_t, \omega_t)\Pr(\omega_t|\omega_0) = \Pi(b_t, y_t|b_t + y_t \leq c_t, \omega_t)[\Pr(\omega_t|\omega_{t-1})]^t \pi_{\omega_0}$

CDF's are moving to the left so that more and more agents are getting lower and lower consumption. Because of the finite grid there is a stationary distribution very skewed to the left. To make the result more clear I start from $\omega_0 = -0.1$.