Econ202B Homework #1 Answer Key Anton Cheremukhin

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Exercise 1 *Occupational Choice in General Equilibrium*

The economy consists of two types of people: measure μ_S skilled and $\mu_U > \mu_S$ unskilled with skill levels S>U>0. Skilled people can either be managers or skilled workers. Unskilled people can only be unskilled workers. Every production unit needs one manager. If a manager manages S skilled and U unskilled workers, the output of that person's production unit is $S^\alpha U^\beta$ where $\alpha + \beta < 1$. The utility function of an individual is: log(c) + $v(h)$ where $h \in \{0, 0.4, 0.7\}$ corresponds to staying at home, working (for both skilled and unskilled) or managing.

a,b) Choose a suitable commodity space for this economy. Define consumption sets, preferences, the aggregate production set and the resource constraint.

c) State the First Welfare Theorem and show that its conditions are satisfied for this problem.

d) Formulate the social planning problem for this economy.

e) What can you say about the relative consumption and relative wages of managers, skilled workers and unskilled workers?

f) Impose conditions on the model parameters that ensure that there are unemployed skilled and unskilled people in equilibrium.

(a) Commodity space $S = R^4 = \{consumption, \pi_{workU}, \pi_{workS}, \pi_{manage}\}$

Consumption sets:

$X_U = \{x_U \in S | \, x_{1U} \geq 0, -1 \leq x_{2U} \leq 0, x_{3U} = 0, x_{4U} = 0\}$

$X_S = \{x_S \in S | \, x_{1S} \geq 0, x_{2S} = 0, x_{3S} \leq 0, x_{4S} \leq 0, -x_{3S} - x_{4S} \leq 1\}$

Preferences:

$U_S(x) = \log(x_{1U}) + x_{2U}(v(0.4) - v(0)) - v(0)$

$U_U(x) = \log(x_{1S}) + x_{3S}(v(0.4) - v(0)) + x_{4S}(v(0.7) - v(0)) - v(0)$

(b) Fraction $-y_2$ are unskilled workers, fraction $-y_3$ are skilled workers, fraction $-y_4$ are managers. Since the technology for each manager exhibits decreasing returns, the workers will be split equally among managers. Hence, each of the $-y_4$ managers produces $\left(\frac{y_2}{y_4}\right)^\beta \left(\frac{y_3}{y_4}\right)^\alpha$ output.

Production set $Y = \left\{y \in S | \, y_2 \leq 0, y_3 \leq 0, y_4 \leq 0, y_1 \leq (-y_2)^\beta (-y_3)^\alpha (-y_4)^{1-\alpha-\beta}\right\}$

Resource constraint $\mu_S x_S + \mu_U x_U = y$

(c) Conditions for the first welfare theorem are: X convex, U - q.concave, no satiation.

By definition of X it is convex. Functions $\log(x)$, ax and bx are strictly increasing - no satiation of $U(.)$. A sum of a strictly concave function $\log(.)$ and a linear function is quasi-concave which accomplishes the proof. Therefore any competitive equilibrium in such an economy is pareto-optimal.

(d) Pareto optimal allocations are allocations (x_S^*, x_U^*, y^*) s.t.:

1) (x_S^*, x_U^*, y^*) is feasible: $x_S^* \in X_S$, $\quad x_U^* \in X_U$, $\quad y^* \in Y$, $\quad x_S^* + x_U^* = y^*$

2) $\nexists \, (x_S', x_U', y')$ s.t.

a)(x_S', x_U', y') is feasible

b)$U_S(x_S') \geq U_S(x_S^*)$ and $U_U(x_U') \geq U_U(x_U^*)$

c)$U_S(x_S') > U_S(x_S^*)$ or $U_U(x^{2\prime}) > U_U(x_U^*)$.

(e) Let's now solve for the pareto-optimal allocation under the assumption of equal weights:
$\mu_S U_S(x_S) + \mu_U U_U(x_U) \to \max_{x_S, x_U, y}$
s.t. $\mu_S x_S + \mu_U x_U = y$
s.t. $x_{1U} \geq 0, -1 \leq x_{2U} \leq 0, x_{3U} = 0, x_{4U} = 0, x_{1S} \geq 0, x_{2S} = 0, x_{3S} \leq 0, x_{4S} \leq 0, -x_{3S} - x_{4S} \leq 1$
s.t. $y_2 \leq 0, y_3 \leq 0, y_4 \leq 0, 0 \leq y_1 \leq (-y_2)^\beta (-y_3)^\alpha (-y_4)^{1-\alpha-\beta}$
This can be simplified to:
$\mu_S U_S(x_S) + \mu_U U_U(x_U) \to \max_{x_S, x_U, y}$
s.t. $\mu_S c_S + \mu_U c_U = y \qquad \mu_U \pi_U = y_U \qquad \mu_S \pi_S = y_S \qquad \mu_S \pi_M = y_M$
s.t. $\pi_U \leq 1, \pi_S + \pi_M \leq 1 \qquad y = (y_U)^\beta (y_S)^\alpha (y_M)^{1-\alpha-\beta}$
Further define: $\qquad (v(0.4) - v(0)) = a \qquad (v(0.7) - v(0)) = b$
$\mu_U (\log(c_U) - a\pi_U) + \mu_S (\log(c_S) - a\pi_S - b\pi_M) \to \max$
s.t. $\mu_S c_S + \mu_U c_U = (\mu_U \pi_U)^\beta (\mu_S \pi_S)^\alpha (\mu_S \pi_M)^{1-\alpha-\beta} = y$
FOC: $\qquad \mu_S \frac{1}{c_S} = \lambda \mu_S \qquad \mu_U \frac{1}{c_U} = \lambda \mu_U$
$\mu_S a = \lambda y \frac{\alpha}{\pi_S} \qquad \mu_S b = \lambda y \frac{1-\alpha-\beta}{\pi_M} \qquad \mu_U a = \lambda y \frac{\beta}{\pi_U}$
Therefore, $c_S = c_U = \frac{y}{\mu_S + \mu_U} = \frac{1}{\lambda}$.
Consumption of all people in the economy is the same.
$\frac{\mu_S}{\mu_S + \mu_U} a = \frac{\alpha}{\pi_S} \qquad \frac{\mu_S}{\mu_S + \mu_U} b = \frac{1-\alpha-\beta}{\pi_M} \qquad \frac{\mu_U}{\mu_S + \mu_U} a = \frac{\beta}{\pi_U}$
$\pi_U = \frac{\beta}{a} \frac{\mu_S + \mu_U}{\mu_U} \qquad \pi_S = \frac{\alpha}{a} \frac{\mu_S + \mu_U}{\mu_S} \qquad \pi_M = \frac{1-\alpha-\beta}{b} \frac{\mu_S + \mu_U}{\mu_S}$
$y = (\mu_U \pi_U)^\beta (\mu_S \pi_S)^\alpha (\mu_S \pi_M)^{1-\alpha-\beta} = (\mu_S + \mu_U) \left(\frac{\alpha}{a}\right)^\alpha \left(\frac{\beta}{a}\right)^\beta \left(\frac{1-\alpha-\beta}{b}\right)^{1-\alpha-\beta}$
$c_S = c_U = \left(\frac{\alpha}{a}\right)^\alpha \left(\frac{\beta}{a}\right)^\beta \left(\frac{1-\alpha-\beta}{b}\right)^{1-\alpha-\beta} = c$
The pareto-optimal allocation is:
$x_U = \left\{c, \frac{\beta}{a} \frac{\mu_S + \mu_U}{\mu_U}, 0, 0\right\}$
$x_S = \left\{c, 0, \frac{\alpha}{a} \frac{\mu_S + \mu_U}{\mu_S}, \frac{1-\alpha-\beta}{b} \frac{\mu_S + \mu_U}{\mu_S}\right\}$
$y = (\mu_S + \mu_U) \left\{c, \frac{\beta}{a}, \frac{\alpha}{a}, \frac{1-\alpha-\beta}{b}\right\}$
To find prices let's solve individual problems:
$\log(c_U) - a\pi_U \to \max \qquad s.t. \qquad p_c c_U - w_U \pi_U = I_M$
FOC: $\qquad \frac{1}{c_U} = \lambda_U p_c \qquad a = \lambda_U w_U$
$\log(c_S) - a\pi_S - b\pi_M \to \max \qquad s.t. \qquad p_c c_S - w_S \pi_S - w_M \pi_M = I_S$
FOC: $\qquad \frac{1}{c_S} = \lambda_S p_c \qquad a = \lambda_S w_S \qquad b = \lambda_S w_M$
From the allocation we know that $c_S = c_U \qquad \Rightarrow \qquad \lambda_S = \lambda_U = 1$ (normalization).
Therefore, $a = w_U = w_S, \qquad w_M = b, \qquad p_c = \frac{1}{c}$.
The wages of skilled and unskilled workers are the same, the wages of managers are higher.
You can arrive to the same conclusion if you use the firm's problem:
$p_c y_U^\alpha y_S^\beta y_M^{1-\alpha-\beta} - w_U y_U - w_S y_S - w_M y_M \to \max$
FOC: $\qquad p_c \frac{\beta y}{y_U} = w_U \qquad p_c \frac{\alpha y}{y_S} = w_S \qquad p_c \frac{(1-\alpha-\beta) y}{y_M} = w_M$
$p_c \frac{\beta c}{\beta / a} = w_U \qquad p_c \frac{\alpha c}{\alpha / a} = w_S \qquad p_c \frac{(1-\alpha-\beta) c}{(1-\alpha-\beta)/b} = w_M$
Normalize $p_c = \frac{1}{c}$, then $\qquad w_U = a \qquad w_S = a \qquad w_M = b$.
(f) For there to be unemployed skilled and unskilled workers we need
$\frac{\beta}{a} \frac{\mu_S + \mu_U}{\mu_U} < 1, \qquad \frac{\alpha}{a} \frac{\mu_S + \mu_U}{\mu_S} + \frac{1-\alpha-\beta}{b} \frac{\mu_S + \mu_U}{\mu_S} < 1$

Exercise 2 *Uncertainty in General Equilibrium*

$U(c,n) = u(c) - v(n) \qquad i = \overline{1,N}$
$u : R^+ \to R, \qquad u \in B(x) \cap C^1(x), \qquad u' > 0, u'' < 0, \qquad u'(0) = \infty$
$v : R^+ \to R, \qquad v \in B(x) \cap C^1(x), \qquad v' > 0, v'' > 0$
$F_j(\epsilon_j, n) = \epsilon_j f(n) \qquad j = \overline{1,M} \qquad \epsilon_j^l \geq 0, \qquad p(l) \geq 0, \qquad \Sigma_{l=1}^L p(l) = 1, \qquad l = \overline{1L}$
$f : R^+ \to R^+, \qquad f \in C^1(x), \qquad f' > 0, f'' < 0, \qquad f'(0) = \infty, f'(\infty) = 0$

(1) According to Debreu commodities are determined by time, location and events.
So the commodity space consists of consumption and labor in each state of nature
$S = \{c^1, ..., c^L, n^1, ..., n^L\} = R^{2L}$
Consumption set $X_i = \{x_i \in S | \, x_i^l \geq 0, x_i^{l+L} \leq 0, l = \overline{1L}\}$
Production set $Y_j = \{y_j \in S | \, y_j^l \geq 0, y_j^{L+l} \leq 0, y_j^l \leq \epsilon_j^l f(-y_j^{L+l}), l = \overline{1L}\}$
$\precsim_i$: $U(x_i) = \Sigma_{l=1}^L p(l)\left[u(x_i^l) - v(-x_i^{L+l})\right]$
$\Theta_j : \theta_{ij} \geq 0, \qquad \Sigma_{i=1}^N \theta_{ij} = 1$
Resource constraint $\Sigma_{i=1}^N x_i \leq \Sigma_{j=1}^M y_j$ $\qquad$ Economy: $E = \left\{(X_i, \precsim_i)_{i=1}^N, (Y_j, \Theta_j)_{j=1}^M, \right\}$

(2) Competitive equilibrium is a tuple: $(x_i^*, y_j^*, p^*) \in R^{2L(M+N+1)}$ such that
a) $x_i^* = \arg\max_{x_i \in X_i} \{U(x_i) | \, p^* x_i \leq \Sigma_{j=1}^M \theta_{ij} p^* y_j^*\}$
b) $y_j^* = \arg\max_{y_j \in Y_j} p^* y_j$ $\qquad$ c) $\Sigma_{i=1}^N x_i^* = \Sigma_{j=1}^M y_i^*$
Conditions for the welfare theorems are:
1stWTh: X_i convex, U - q.concave, no satiation.
2ndWTh: X_i convex, finite demetsional, U - continuous, q.concave, no satiation, Y_j convex.

By definition of X_i it is convex. Taking into account that $f(.)$ is strictly increasing and strictly concave and by definition of Y_j it is convex. Functions $u(.)$ and $v(.)$ being continuously differentiable implies continuity, and being strictly increasing - no satiation of $U(.)$. Moreover a sum of strictly concave functions $u(.)$ and $-v(.)$ is strictly quasi-concave which accomplishes the proof.

Since X_i is bounded below, closed and connected, preferences are continuous, Y_j is closed and convex, and gives no free lunch the optimum exists. Since X_i is non-empty, bounded below, closed and convex, preferences are continuous, quasi-concave and exhibit no satiation, Y_j is closed and convex, production allows inaction, is irreversible and allows free disposal the equilibrium also exists. Therefore, we can first characterize the optimum which will coincide with equilibrium allocations.

(3) $\max_{x_i \in X_i, y_j \in Y_j} \left\{\Sigma_{i=1}^N \phi_i \Sigma_{l=1}^L p(l)\left[u(x_i^l) - v(-x_i^{L+l})\right] \Big| \qquad \Sigma_{i=1}^N x_i \leq \Sigma_{j=1}^M y_j, \qquad y_j^l \leq \epsilon_j^l f(-y_j^{L+l})\right\}$

Let's substitute (for the sake of simplicity) c_i^l and n_i^l instead of x_i^l and $-x_i^{l+L}$ for consumers and n_j^l instead of $-y_j^{l+L}$ for producers. Then the Lagrangian is:

$$\max_{c_i^l, n_i^l, n_j^l} \Sigma_{l=1}^L \left(p(l)\Sigma_{i=1}^N \phi_i\left[u(c_i^l) - v(n_i^l)\right] + \lambda^l\left[\Sigma_{i=1}^N n_i^l - \Sigma_{j=1}^M n_j^l\right] + \mu^l\left[\Sigma_{j=1}^M \epsilon_j^l f(n_j^l) - \Sigma_{i=1}^N c_i^l\right]\right)$$

This leads to the following first order conditions:

$$\phi_i p(l) u'(c_i^l) = \mu^l \qquad \phi_i p(l) v'(n_i^l) = \lambda^l \qquad \lambda^l = \mu^l \epsilon_j^l f'(n_j^l) \Rightarrow \qquad \frac{\lambda^l}{\mu^l} = \frac{v'(n_i^l)}{u'(c_i^l)} = \epsilon_j^l f'(n_j^l)$$

$$\phi_i p(l) u'(c_i^l) = \mu^l = \phi_{-i} p(l) u'(c_{-i}^l) \qquad \phi_i p(l) v'(n_i^l) = \lambda^l = \phi_{-i} p(l) v'(n_{-i}^l)$$

The assumption of CES utility functions for both consumption and labor ensure that the consumption and labor sharing rules are linear:

$$\phi_i (c_i^l)^{-\sigma} = \phi_{-i}(c_{-i}^l)^{l-\sigma}, \phi_i (n_i^l)^{-\rho} = \phi_{-i}(n_{-i}^l)^{-\rho} \qquad \Rightarrow \qquad c_i^l = \left(\frac{\phi_i}{\phi_{-i}}\right)^{1/\sigma} c_{-i}^l, n_i^l = \left(\frac{\phi_i}{\phi_{-i}}\right)^{1/\rho} n_{-i}^l$$

Another possible assumption is that the central planner assigns equal weights to all consumers ($\phi_i = \phi_{-i}$), than all consumers will be offered identical behavior independent of the form of their utility function: $n_i^l = n_{-i}^l \qquad c_i^l = c_{-i}^l$.In this case labor and consumption will be equally shared between consumers in each state. However the formula above for the sharing rules is true in both cases. We can use it to find the solution:

$$n_{i_0}^l \phi_{i_0}^{-1/\sigma} \Sigma_{i=1}^N \phi_i^{1/\sigma} = \Sigma_{j=1}^M n_j^l \qquad c_{i_0}^l \phi_{i_0}^{-1/\rho} \Sigma_{i=1}^N \phi_i^{1/\rho} = \frac{1}{N} \Sigma_{j=1}^M \epsilon_j^l f\left(n_j^l\right)$$

$$\frac{v'\left(n_{i_0}^l\right)}{u'\left(c_{i_0}^l\right)} = \frac{v'\left(\left[\Sigma_{j=1}^M n_j^l\right] / \left[\phi_{i_0}^{-1/\rho} \Sigma_{i=1}^N \phi_i^{1/\rho}\right]\right)}{u'\left(\left[\Sigma_{j=1}^M \epsilon_j^l f\left(n_j^l\right)\right] / \left[\phi_{i_0}^{-1/\sigma} \Sigma_{i=1}^N \phi_i^{1/\sigma}\right]\right)} = \epsilon_j^l f'\left(n_j^l\right), \qquad j = \overline{1, M}, l = \overline{1, L}, i_0 = \overline{1, N}$$

So we have (2N+M)L equations and (2N+M)L unknowns. The solution is unique sinse l.h.s. of the last equation is strictly increasing in each argument n_j^l while the r.h.s. is strictly decreasing $(v'' > 0, u'' < 0, f' > 0, f'' < 0)$. Knowing this unique solution we could solve backwards to find equilibrium consumption and labor. This gives us $L\left(2N + M\right)$ equilibrium values.

In general we get $L(M + 2N + 2)$ equations with $L\left(M + 2N + 2\right)$ unknowns which due to Inada conditions have a unique solution: $\quad \{n_j^l, n_i^l, c_i^l, \lambda^l, \mu^l\} \qquad \Sigma_{i=1}^N n_i^l = \Sigma_{j=1}^M n_j^l$

$$\phi_i p\left(l\right) u'\left(c_i^l\right) = \mu^l \qquad \phi_i p\left(l\right) v'\left(n_i^l\right) = \lambda^l \qquad \lambda^l = \mu^l \epsilon_j^l f'\left(n_j^l\right) \qquad \Sigma_{j=1}^M \epsilon_j^l f\left(n_j^l\right) = \Sigma_{i=1}^N c_i^l$$

(4)

$$L\left(x_i\right) = \Sigma_{l=1}^L p\left(l\right)\left[u\left(c_i^l\right) - v\left(n_i^l\right)\right] + \lambda_i \Sigma_{l=1}^L \left[\Sigma_{j=1}^M \theta_{ij}\left(p_c^l y_j^l - p_n^l n_j^l\right) + p_n^l n_i - p_c^l c_i\right]$$

FOC: $\quad p\left(l\right) u'\left(c_i^l\right) = p_c^l \lambda_i \qquad p\left(l\right) v'\left(n_i^l\right) = p_n^l \lambda_i \qquad\qquad (N * L * 2$ equations$)$

Restating gives: $\frac{p(l) u'\left(c_i^l\right)}{p_c^l} = \lambda_i = \frac{p(l) v'\left(n_i^l\right)}{p_n^l}$ which implies:

$$p_c^l = p_c^1 \frac{p(l) u'\left(c_i^l\right)}{p(1) u'\left(c_i^1\right)} \qquad p_n^1 = p_c^1 \frac{v'\left(n_i^1\right)}{u'\left(c_i^1\right)} \qquad p_n^l = p_n^1 \frac{p(l) u'\left(n_i^l\right)}{p(1) u'\left(n_i^1\right)} = p_c^1 \frac{v'\left(n_i^1\right)}{u'\left(c_i^1\right)} \frac{p(l) u'\left(n_i^l\right)}{p(1) u'\left(n_i^1\right)}$$

Putting the price of consumption in state 1 $p_c^1 = 1$ assigns a numerair, which determines the rest of the prices.

In general in the competitive setting an analog of central planner's weights are the shares in firms outputs. The budget constraints give us N equations:

$$\Sigma_{l=1}^L \Sigma_{j=1}^M \theta_{ij}\left(p_c^l y_j^l - p_n^l n_j^l\right) = \Sigma_{l=1}^L \left[p_c^l c_i - p_n^l n_i\right]$$

From the firm's problem it follows that $\qquad \epsilon_j^l f'\left(n_j^l\right) = \frac{p_n^l}{p_c^l} \qquad (M * L$ constraints$)$

Resource constraints imply: $\qquad \Sigma_{j=1}^M \epsilon_j^l f\left(n_j^l\right) = \Sigma_{i=1}^N c_i^l \qquad \Sigma_{i=1}^N n_i^l = \Sigma_{j=1}^M n_j^l$ $(2 * L$ constraints$)$

The whole system has $L\left(M + 2N + 2\right)$ equations with the same number of unknowns, and due to Inada conditions has a unique solution. The only free parameter left is the choice of numeraire price. It is easy to show that $\lambda^l = p_n^l$ and $\mu^l = p_c^l$ is one of the price solutions. In this case the numeraire is specified by the fact that probabilities of states sum up to one. In the competitive setting the sharing rules depend on the distribution of wealth. An analogous situation of linear sharing rules arizes if all agents have equal wealth, i.e. equal shares in the firm's profits:

$$\theta_{ij} = \frac{1}{N} \qquad \Rightarrow \qquad n_i^l = n_{-i}^l \equiv n^l \qquad c_i^l = c_{-i}^l \equiv c^l$$

Exercise 3 *Private Information in General Equilibrium*

There is a measure one of ex ante identical consumers. First all the decisions are made, then uncertainty is realized. With probability $p = \frac{1}{2}$ each consumer becomes type I and with probability $(1-p) = \frac{1}{2}$ she becomes type II. Type I consumers are risk-averse: $u_1(c) = \sqrt{c}$, while type II consumers are risk-neutral: $u_2(c) = c$. The endowment is $e = 1$ per person. Types are private information. Specify the economy in the language of Debreu using a grid $C=\left\{0, \frac{1}{4}, c_{\max}\right\}$. Describe the timing of the economy. Specify $c_{\max}$ such that the first best allocation can be achieved in general equilibrium. Find an allocation and price vector.

Planner's problem (general version):

$\max_{x_n^i} \Sigma_i \frac{1}{2} \Sigma_n x_n^i u_i(c_n)$

s.t. $\quad x_n^i \geq 0, \quad \Sigma_n x_n^i = 1.\ \forall i \forall n$

s.t. $\quad \Sigma_i \frac{1}{2} \Sigma_n x_n^i c_n \leq 1 \quad$ (resource constraint)

s.t. $\quad \Sigma_n x_n^i u_i(c_n) \geq \Sigma_n x_n^j u_i(c_n) \quad \forall i, \forall j \quad$ (incentive compartibility constraint)

a) In our case $i = \{1, 2\}$, $n = \{1, 2, 3\}$, $c_n = \left\{0, \frac{1}{4}, c_{\max}\right\}$.

Therefore, $\max_{x_n^i} \frac{1}{2}\left(\frac{1}{4}x_2^1 + c_{\max}x_3^1\right) + \frac{1}{2}\left(\frac{1}{2}x_2^2 + \sqrt{c_{\max}}x_3^2\right)$

s.t. $\quad x_n^i \geq 0, \quad \Sigma_n x_n^i = 1.\ \forall i$

s.t. $\quad \frac{1}{2}\left(\frac{1}{4}x_2^1 + c_{\max}x_3^1\right) + \frac{1}{2}\left(\frac{1}{4}x_2^2 + c_{\max}x_3^2\right) \leq 1$

s.t. $\quad \frac{1}{2}x_2^2 + \sqrt{c_{\max}}x_3^2 \geq \frac{1}{2}x_2^1 + \sqrt{c_{\max}}x_3^1$

s.t. $\quad \frac{1}{4}x_2^1 + c_{\max}x_3^1 \geq \frac{1}{4}x_2^2 + c_{\max}x_3^2$

In the language of Debreu: $\quad S = R^{2\times 2} = \left\{\pi^1_{\frac{1}{4}}, \pi^1_{c_{\max}}, \pi^2_{\frac{1}{4}}, \pi^2_{c_{\max}}\right\}$

Consumption set:

$X = \left\{x \in S | x_{n=2,3}^{i=1,2} \geq 0, \Sigma_n x_n^i \leq 1, \frac{1}{2}x_2^2 + \sqrt{c_{\max}}x_3^2 \geq \frac{1}{2}x_2^1 + \sqrt{c_{\max}}x_3^1, \frac{1}{4}x_2^1 + c_{\max}x_3^1 \geq \frac{1}{4}x_2^2 + c_{\max}x_3^2\right\}$

Preferences: $\quad U(x) = \frac{1}{2}\left(\frac{1}{4}x_2^1 + c_{\max}x_3^1\right) + \frac{1}{2}\left(\frac{1}{2}x_2^2 + \sqrt{c_{\max}}x_3^2\right)$

Production set: $\quad Y = \left\{y \in S | y_{n=2,3}^{i=1,2} \geq 0, \frac{1}{2}\left(\frac{1}{4}y_2^1 + c_{\max}y_3^1\right) + \frac{1}{2}\left(\frac{1}{4}y_2^2 + c_{\max}y_3^2\right) \leq 1\right\}$

Resource constraint: $x = y$.

First commodities are traded (=contracts are signed), then types become known to the consumers, then the planner asks the consumers about their types, and using the message assigns each consumer an allocation according to the initial contract. Contracts are signed first, because otherwise, after knowing the types some of the agents would prefer to remain in autarky, but they can't walk away from pre-assigned contracts.

If $\frac{1}{2}x_2^2 + \sqrt{c_{\max}}x_3^2 = \frac{1}{2}x_2^1 + \sqrt{c_{\max}}x_3^1$ then $\frac{1}{4}x_2^1 + c_{\max}x_3^1 > \frac{1}{4}x_2^2 + c_{\max}x_3^2$, therefore the second IC constraint is not binding.

$\max_{x_n^i} \frac{1}{2}\left(\frac{1}{4}x_2^1 + c_{\max}x_3^1\right) + \frac{1}{2}\left(\frac{1}{2}x_2^2 + \sqrt{c_{\max}}x_3^2\right)$

s.t. $\quad \frac{1}{4}x_2^1 + c_{\max}x_3^1 + \frac{1}{4}x_2^2 + c_{\max}x_3^2 = 2$

s.t. $\quad \frac{1}{2}x_2^2 + \sqrt{c_{\max}}x_3^2 = \sqrt{c_{\max}}x_3^1$

To discriminate between the two consumers the planner needs to deliver a fixed amount to the risk-averse consumer and a lottery to the risk-neutral consumer: $x_3^2 = 0, x_2^2 = 1, x_2^1 = 0$.

$\max_{x_n^i} \frac{1}{2}\left(c_{\max}x_3^1 + \frac{1}{2}\right)$

s.t. $\quad c_{\max}x_3^1 + \frac{1}{4} = 2 \qquad$ s.t. $\quad \frac{1}{2} = \sqrt{c_{\max}}x_3^1$

The first two equations define the Maximum Welfare: $W = \frac{1}{2}\left(2 - \frac{1}{4} + \frac{1}{2}\right) = \frac{9}{8}$

It is independent of $c_{\max}$ (only need to make sure that $x_3^1 \leq 1$).

$\left\{c_{\max}x_3^1 = \frac{7}{4} \quad \frac{2}{4} = \sqrt{c_{\max}}x_3^1\right\} \quad \Rightarrow \sqrt{c_{\max}} = \frac{7}{2} \quad c_{\max} = \frac{49}{4} = 12.25 \quad x_3^1 = \frac{1}{7}$

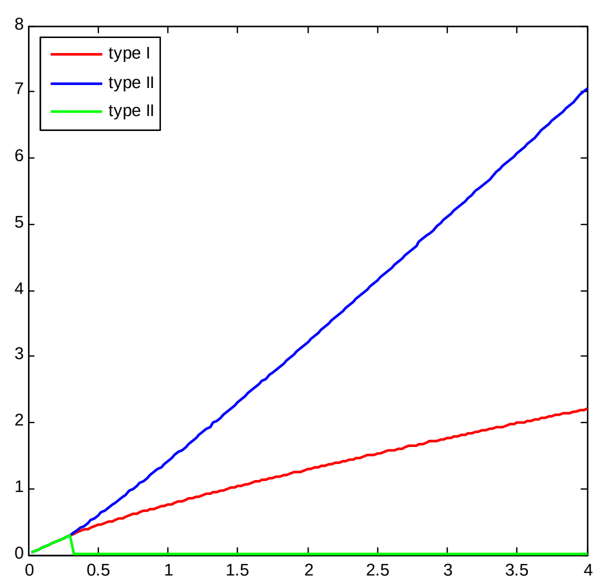
The equilibrium allocation therefore is: $\quad \left\{0, \frac{1}{7}, 1, 0\right\}$.

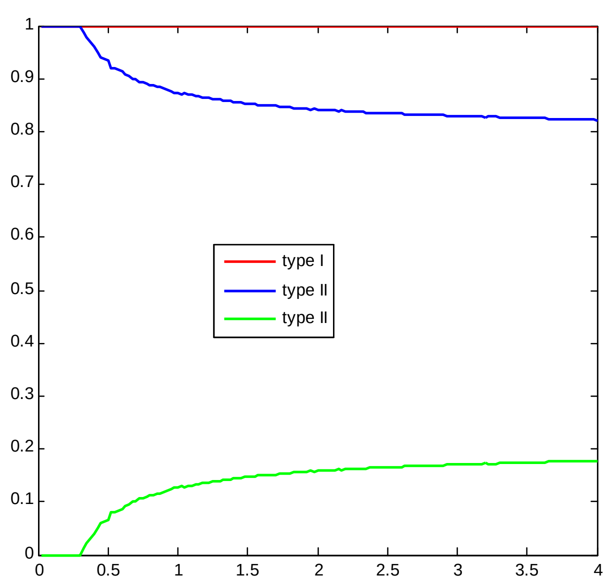
Use firm profit maximization to find prices:

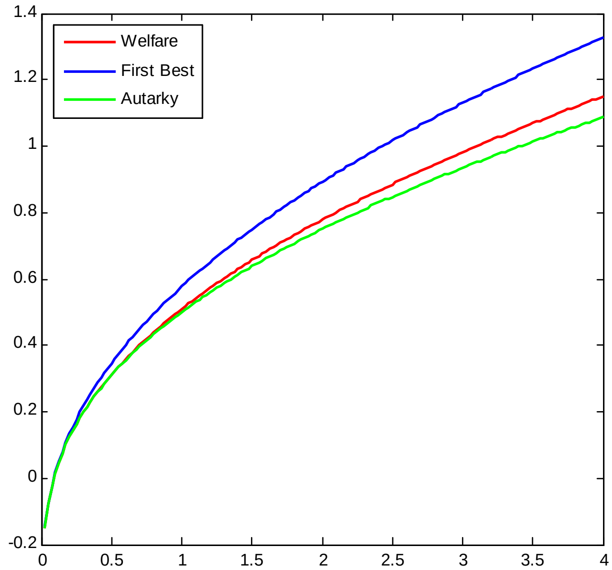
$L = p_2^1 y_2^1 + p_3^1 y_3^1 + p_2^2 y_2^2 + p_3^2 y_3^2 + \kappa\left(2 - \left(\frac{1}{4}y_2^1 + c_{\max} y_3^1\right) - \left(\frac{1}{4}y_2^2 + c_{\max} y_3^2\right)\right)$
$p_2^1 \leq \frac{1}{4}\kappa \qquad p_3^1 \leq c_{\max}\kappa \qquad p_2^2 \leq \frac{1}{4}\kappa \qquad p_3^2 \leq c_{\max}\kappa$ For example, $p = \{1, 49, 1, 49\}$.
The consumer problem would have a lot of constraints, you can find multipliers that would fit.

Exercise 4 *Private Information*

$\max_{x_n^i} \frac{1}{2}\Sigma_n x_n^1 \frac{1}{8}\log c_n + \frac{1}{2}\Sigma_n x_n^2 \sqrt{c_n}$
s.t. $x_n^i \geq 0, \qquad \Sigma_n x_n^i = 1.\ \forall i \forall n$
s.t. $\Sigma_n x_n^1 c_n + \Sigma_n x_n^2 c_n \leq 2e$ (RC)
s.t. $\Sigma_n x_n^1 \frac{1}{8}\log(c_n) \geq \Sigma_n x_n^2 \frac{1}{8}\log(c_n)$ (IC_1)
s.t. $\Sigma_n x_n^2 \sqrt{c_n} \geq \Sigma_n x_n^1 \sqrt{c_n}$ (IC_2)
To solve that on the computer we need to state it in terms of constraints:
Let $c = [c_1, ..., c_n]$
let $U_1 = \frac{1}{8}\log c, \qquad U_2 = \sqrt{c}$
$f = \left[-\frac{1}{2}U_1, -\frac{1}{2}U_2\right];$
$A = [c, c; -U_1, U_1; U_2, -U_2];$
$b = [2e; 0; 0];$
$Aeq = [ones(1,n), zeros(1,n); zeros(1,n), ones(1,n)];$
$beq = [1; 1];$
$ub = ones(1, 2n);$
$lb = zeros(1, 2n);$
linprog(f,A,b,Aeq,beq,lb,ub)
The result is a single values for type I and a pair of values for type II as a function of the endowment. For low values of endowment the economy remains in autarky. The functions and probabilities are depicted in the graph:






Welfare
First Best
Autarky
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.5
1
1.5
2
2.5
3
3.5
4