Econ202B Midterm Answer Key Anton Cheremukhin

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Exercise 1 *Dynamic Contracting*

Consumer: $\Sigma_s \Sigma_{t=0}^{\infty} \beta^t \pi_s \ln(y_{st} + \tau_{st})$ $s.t.$ $c_t = y_t + \tau_t$ $y_t = \begin{cases} y_L, & \pi_L \\ y_H, & \pi_H \end{cases}$

Planner: $\Sigma_s \Sigma_{t=0}^{\infty} \pi_s \gamma^t (-\tau_{st})$ $\gamma < \beta$

(a) Public info: $\Sigma_s \Sigma_{t=0}^{\infty} \gamma^t \pi_s (-\tau_{st}) \to \max_{\tau_t}$ s.t. $\Sigma_s \Sigma_{t=0}^{\infty} \beta^t \pi_s \ln(y_{st} + \tau_{st}) \geq w_0$

$\Sigma_s \Sigma_{t=0}^{\infty} \gamma^t \pi_s (-\tau_{st}) + \lambda (\Sigma_s \Sigma_{t=0}^{\infty} \beta^t \pi_s \ln(y_{st} + \tau_{st}) - w_0) \to \max_{\tau_t}$

$\gamma^t \pi_s = \lambda \beta^t \pi_s \frac{1}{y_{st}+\tau_{st}}$ $\Rightarrow$ $\frac{\gamma^t}{\beta^t} = \frac{\lambda}{y_{st}+\tau_{st}}$ $\Rightarrow$ consumption grows at a constant rate:

$\frac{\lambda}{y_{st}+\tau_{st}} \frac{\gamma}{\beta} = \frac{\gamma^{t+1}}{\beta^{t+1}} = \frac{\lambda}{y_{st+1}+\tau_{st+1}}$ $\Rightarrow$ $\frac{c_{st+1}}{c_{st}} = \frac{\beta}{\gamma} > 1.$

$w_0 = \Sigma_s \Sigma_{t=0}^{\infty} \beta^t \pi_s \ln(c_t) = \Sigma_{t=0}^{\infty} \beta^t \ln\left(c_0 \left(\frac{\beta}{\gamma}\right)^t\right) = \Sigma_{t=0}^{\infty} \beta^t \left[\ln c_0 + t \ln \frac{\beta}{\gamma}\right] = \frac{\ln c_0}{1-\beta} + \frac{\beta}{(1-\beta)^3} \ln \frac{\beta}{\gamma}$

Since, $\sum_{t=0}^{\infty} \beta^t = \frac{1}{1-\beta}$ $\sum_{t=0}^{\infty} t\beta^t = \frac{\beta}{(1-\beta)^3}$ $\Rightarrow$ $c_0 = \exp\left[(1-\beta) w_0 - \frac{\beta}{1+\beta+\beta^2} \ln \frac{\beta}{\gamma}\right]$

$c_t = c_0 \left(\frac{\beta}{\gamma}\right)^t = \exp\left[(1-\beta) w_0 - \frac{\beta}{1+\beta+\beta^2} \ln \frac{\beta}{\gamma} + \frac{\beta}{\gamma} t\right]$

Consumption is increasing as time goes. It is independent of income, so there is full insurance.

(b) Private info: $P(v) = \max_{\tau_s, w_s} \Sigma_s \pi_s [-\tau_s + \gamma P(w_s)]$

s.t. $\Sigma_s \pi_s (\ln(y_s + \tau_s) + \beta w_s) = v$

s.t. $\ln(y_s + \tau_s) + \beta w_s \geq \ln(y_s + \tau_k) + \beta w_k$

The agents first sign contracts they can't walk away from, then uncertainty is realized, then they report their income realization, then contracts are enforced. The truth-telling contraint insures that agents don't misreport their incomes. This is the only difference from part (a), where there is no need for reports since realizations are public information.

$L = \Sigma_s \pi_s [-\tau_s + \gamma P(w_s)] + \lambda (v - \Sigma_s \pi_s (\ln(y_s + \tau_s) + \beta w_s)) - ...$

$-\mu (\ln(y_H + \tau_H) + \beta w_H - \ln(y_H + \tau_L) - \beta w_L)$

$-\pi_H - \frac{\lambda \pi_H}{y_H+\tau_H} - \frac{\mu}{y_H+\tau_H} = 0$ $\quad -\pi_L - \frac{\lambda \pi_L}{y_L+\tau_L} + \frac{\mu}{y_H+\tau_L} = 0$

$\pi_H \gamma P'(w_H) - \lambda \pi_H \beta - \mu\beta = 0$ $\quad \pi_L \gamma P'(w_L) - \lambda \pi_L \beta + \mu\beta = 0$

Envelope: $P'(v) = \lambda$ $\Rightarrow$ $\pi_H (\gamma P'(w_H) - \lambda\beta) = \mu\beta$ $\quad \pi_L (\gamma P'(w_L) - \lambda\beta) = -\mu\beta$

$\gamma \Sigma_s \pi_s P'(w_s) = \beta\lambda = \beta P'(v)$ $\Leftrightarrow$ $E_t P'(v_{t+1}) = \frac{\beta}{\gamma} P'(v_t) < P'(v_t) < 0$

Since $P'(w)$ is a decreasing function, the promised utility will be increasing over time, which means that the expected marginal cost of delivering insurance will be higher and higher over time and expected consumption will be increasing. However, variance of consumption would also be increasing since utility is concave and a higher difference in consumption would be needed to maintain the same difference in marginal utilities to make sure that rich agents tell the truth. I have no idea which of the forces dominates and whether $P'(v) \underset{a.s.}{\to} \infty$ or $P'(v) \underset{a.s.}{\to} 0$ or there is a stationary distribution.

Exercise 2 *Externalities in General Equilibrium*

(a) $S = R^3 = \{c, n_{SN}, n_{NS}\}$ $\quad n_{SS} = S - n_{SN}$ $\quad n_{NN} = N - n_{NS}$
$X_S = \{x_S \in S | x_{S1} \geq 0, 0 \leq x_{S2} \leq 1, x_{S3} = 0\}$
$U_S(x_S) = \ln(x_{S1}) - x_{S2} v(1) - (1 - x_{S2}) v(2)$
$X_N = \{x_N \in S | x_{N1} \geq 0, x_{N2} = 0, 0 \leq x_{N3} \leq 1\}$
$U_N(x_N) = \ln(x_{N1}) - x_{N3} v(1) - (1 - x_{N3}) v(0)$
$Y = \{y \in S | y_1 \leq 1, y_2 = y_3 \geq 0\}$
$Sx_S + Nx_N = y$ $\quad$ type i owns θ_i share of the firm.
(b) Equilibrium: $\quad (x_S^*, x_N^*, y^*, p^*)$ s.t.
1) $x_i^* \in X_i, y \in Y, Sx_S + Nx_N = y$
2) $x_i^* = \arg\max_{x_i \in X_i} U_i(x_i) | p^* x_i \leq \theta_i p^* y^*$
3) $y^* = \arg\max_{y \in Y} p^* y$
(c) Solve for Pareto-optimum
$S(\ln(x_{S1}) + a x_{S2}) + N(\ln(x_{N1}) + b x_{N3}) \to \max_x$
s.t. $\quad Sx_{S2} = Nx_{N3}$ $\quad$ s.t. $\quad (1 - Sx_{S1} - Nx_{N1}) = 0$
FOC: $\quad \frac{S}{Sx_{S1}} = \frac{N}{Nx_{N1}} \quad \Rightarrow \quad x_{S1} = x_{N1} = 1$
The problem simplifies to:
$Sax_{S2} + Nbx_{N3} \to \max_x \quad s.t. \quad Sx_{S2} = Nx_{N3} \quad \Rightarrow \quad (a + b) Sx_{S2} \to \max_x$
Since the function is linear, we shall have a corner solution.
If $a + b < 0$ all smokers and nonsmokers will be sorted into different restaurants.
If $a + b > 0$ smokers will be mixed with nonsmokers.
This is equivalent to: $v(2) - v(1) > v(1) - v(0)$. Preferences should be convex.
In this case $x_{2S} = 1$ (all nonsmokers are mixed with smokers).
Hence, $x_{3N} = \frac{S}{N} x_{2S}$. Since $S < N$ some smokers will be matched with smokers.
Now to find the prices solve for equilibrium.
Smokers:
$U_S(x_S) = \ln(x_{S1}) + x_{S2}(v(2) - v(1)) = \ln(x_{S1}) + a x_{S2} \to \max$
s.t. $x_{S1} + p_2 x_{S2} \leq I_S \quad \Rightarrow \quad \ln(x_{S1}) + \frac{a}{p_2}(I_S - x_{S1}) \to \max_{x_{S2}}$
FOC: $\quad p_2 \leq a x_{S1} = v(2) - v(1) > 0$ $\quad$ (Choose a corner solution).
Nonsmokers:
$U_N(x_N) = \ln(x_{N1}) + x_{N3}(v(0) - v(1)) = \ln(x_{N1}) + b x_{N3} \to \max$
s.t. $x_{N1} + p_3 x_{N3} \leq I_N \quad \Rightarrow \quad \ln(x_{N1}) + \frac{b}{p_3}(I_N - x_{N1}) \to \max_{x_{N3}}$
FOC: $\quad p_3 = b x_{N1} = v(0) - v(1) < 0$ $\quad$ (Choose an interior solution).
Firm:
$y_1 + p_2 y_2 + p_3 y_3 + \lambda(1 - y_1) + \mu(y_2 - y_3) \to \max_y$
$1 = \lambda \quad - p_2 = \mu = p_3$
In equilibrium the nonsmokers will have an interior solution. Therefore,
$p_3 = v(0) - v(1) \quad p_2 = v(1) - v(0) < v(2) - v(1)$
$x_S = \{1, 1, 0\} \quad x_N = \left\{1, 0, \frac{S}{N}\right\} \quad y = \{1, S, S\} \quad p = \{1, v(1) - v(0), v(0) - v(1)\}$
(d) Allocations reached in equilibrium are pareto-optimal, so there are no inefficiencies in equilibrium. If agents increasingly dislike smoke with the number of smokers present, then all the smokers will be matched with nonsmokers. Nonsmokers will be compensated by smokers for bearing smoke in the restaurant. This is reflected in the negative price paid by nonsmokers (they get paid). If preferences are concave, no smokers will be mixed with nonsmokers and no compensation is required.