Economics 202B, Answer Key for the Final Exam Anton Cheremukhin

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Exercise 1 *General Equilibrium*

(a) $S = R^6 = \{c_1, c_2, c_4, \pi_1^{emp}, \pi_2^{emp}, \pi_4^{emp}\}$ (2.5 pt)
(b) $X = \{x \in S | x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, -x_4 \in [0,1], -x_5 \in [0,1], -x_6 \in [0,1]\}$ (2.5 pt)
(c) $u(x) = (\log x_1 + 3x_4) + \frac{3}{5}\beta(\log x_2 + 3x_5) + \frac{2}{5}\beta(\log x_3 + 3x_6)$ (2.5 pt)
(d) $Y = \{y \in S | y_1 \leq -y_4, y_2 \leq -2y_5, y_3 \leq -4y_6, y_4 \leq 0, y_5 \leq 0, y_6 \leq 0\}$ (2.5 pt)
$RC: \quad x = y$
All consumers are the same, so we can think of one.
(e) Allocation (x, y) is a pareto-optimum (PO) iff: (5 pt)
1) $x = y, x \in X, y \in Y$
2) $\nexists (\tilde{x}, \tilde{y}) : \tilde{x} = \tilde{y}, \tilde{x} \in X, \tilde{y} \in Y$, and $u(\tilde{x}) > u(x)$
(f) A competitive equilibrium is a tuple: (x^*, y^*, p^*) s.t. (5 pt)
1) $x^* = \arg\max_{x \in X} \{u(x) | p^* x \leq 0\}$
2) $y^* = \arg\max_{y \in Y} p^* y$
3) $x^* - y^* = 0$

(g) First welfare theorem holds because the consumption set is convex, preferences are quasi-concave and there is no satiation. So we can start with the solution of PO problem: $\max_{y \in Y} u(y)$

$(\log y_1 - 3y_1) + \frac{3}{5}\beta\left(\log y_2 - 3\frac{y_2}{2}\right) + \frac{2}{5}\beta\left(\log y_3 - 3\frac{y_3}{4}\right) \to \max_y$
FOCs: $\frac{1}{y_1} = 3 \quad \frac{1}{y_2} = \frac{3}{2} \quad \frac{1}{y_3} = \frac{3}{4}$

$$\boxed{x^* = \left(\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right) = y^*} \quad \text{(5 pt)}$$

Prices: $\max_{x \in X} u(x) - \lambda p x$
FOCs: $u_i'(x) = \lambda p_i$
$u'(x) = \left[\frac{1}{x_1}, \frac{3}{5}\beta\frac{1}{x_2}, \frac{2}{5}\beta\frac{1}{x_3}, 3, \frac{9}{5}\beta, \frac{6}{5}\beta\right]$ let $\lambda = \frac{10}{3}$ be the normalization.

$$\boxed{p = (10, 3\beta, \beta, 10, 6\beta, 4\beta)} \quad \text{(5 pt)}$$

Exercise 2 *Ramsey Taxation*

$$g_0 = 0 \qquad g_1 = \begin{cases} 0 & \pi \\ \frac{1}{2} & 1-\pi \end{cases} \qquad b_0 = 0 \qquad c_i + g_i = n_i$$

$U = u(c_0) - n_0 + E[\log(c_1) - n_1]$

(a) The consumer's problem is to maximize U given the constraints:

$c_0 + q_1 b_1 + q_G b_G = (1-\tau_0) n_0$

$c_1 = (1-\tau_1) n_1 + b_1$

$c_G = (1-\tau_G) n_G + b_G$

Therefore, $c_0 + q_1(c_1 - (1-\tau_1)n_1) + q_G(c_G - (1-\tau_G)n_G) = (1-\tau_0)n_0$

Hence, $\log(c_0) - n_0 + \pi[\log(c_1) - n_1] + (1-\pi)[\log(c_G) - n_G] \to \max_{n_i, c_i}$ (5 pt)

s.t. $c_0 + q_1(c_1 - (1-\tau_1)n_1) + q_G(c_G - (1-\tau_G)n_G) = (1-\tau_0)n_0$

FOC: $\frac{1}{c_0} = \lambda \qquad \pi\frac{1}{c_1} = q_1\lambda \qquad (1-\pi)\frac{1}{c_G} = q_G\lambda$

FOC: $\lambda(1-\tau_0) = 1$

FOC: $\lambda q_1 (1-\tau_1) = \pi$

FOC: $\lambda q_G (1-\tau_G) = (1-\pi)$

In other words: $\boxed{\frac{1-\tau_i}{c_i} = 1}$. (5 pt)

(b) The implementability condition is then:

$\frac{1}{c_0}(c_0 - (1-\tau_0)n_0) + \pi\frac{1}{c_1}(c_1 - (1-\tau_1)n_1) + (1-\pi)\frac{1}{c_G}(c_G - (1-\tau_G)n_G) = 0$

$\boxed{n_0 + \pi n_1 + (1-\pi)n_G = 2} = 1 + \pi + (1-\pi)$ (5 pt)

(c) The Ramsey plan maximizes welfare subject to the implementability constraint and the three resource constraints:

$$L = \begin{bmatrix} [u(c_0) + \Phi] + \pi[u(c_1) + \Phi] + \\ +(1-\pi)[u(c_G) + \Phi] + [-n_0 - \Phi n_0] + \\ +\pi[-n_1 - \Phi n_1] + (1-\pi)[-n_G - \Phi n_G] + \\ +\psi_0(n_0 - c_0 - g_0) + \pi\psi_1(n_1 - c_1 - g_1) + (1-\pi)\psi_G(n_G - c_G - g_G) \end{bmatrix}$$

FOC: $\psi_i = \boxed{1 + \Phi = \frac{1}{c_i}}$ (5 pt)

The Ramsey problem solves four equations for $\{n_0, n_1, n_G, \Phi\}$:

$$2 = n_0 + \pi n_1 + (1-\pi)n_G$$
$$1 + \Phi = \frac{1}{n_i - g_i}$$

(d) From the first-order condition: $1 - \tau_i = n_i - g_i = \frac{1}{1+\Phi} = c_i \quad \Rightarrow \quad \tau_i = \frac{\Phi}{1+\Phi}$ (3 pt)

So all the taxes are the same! Consumption in all periods is the same. Labor input is different.

Hence, $n_i = \frac{1}{1+\Phi} + g_i \quad \Rightarrow \quad n_0 = \frac{1}{1+\Phi} = n_1, \qquad n_G = \frac{1}{1+\Phi} + \frac{1}{2}$

$2 = n_0 + \pi n_1 + (1-\pi)n_G \quad \Rightarrow$

$$\frac{\Phi}{1+\Phi} = \boxed{\tau = \frac{1}{4}(1-\pi)} \quad \text{(1 pt)}$$

$$\boxed{n_0 = n_1 = 1 - \tau} \qquad \boxed{n_G = \frac{3}{2} - \tau} \quad \text{(2 pt)}$$

$$\boxed{b_1} = \tau_1 n_1 = \boxed{\tau(1-\tau)} \qquad \boxed{b_G} = \tau_G n_G - g_G = \boxed{\tau\left(\frac{3}{2} - \tau\right) - \frac{1}{2}} \quad \text{(4 pt)}$$

$$q_1 b_1 + q_G b_G = -\tau_0 n_0 = -\tau(1-\tau)$$

Exercise 3 *Externality in production of human capital*

Planner's problem:

$\Sigma_{t=0}^{\infty}\beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \rightarrow \max_{\{c_t,k_{t+1},h_{t+1}\}}$ s.t. $c_t + k_{t+1} - (1-\delta)k_t \leq Ak_t^{\alpha}(h_{1t})^{1-\alpha}$

s.t. $h_{t+1} = Bh_{2t}^{\phi}(h_t)^{\psi}$ s.t. $h_t = h_{1t} + h_{2t}$

(a) Consumer's problem: (4 pt)

$\Sigma_{t=0}^{\infty}\beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \rightarrow \max_{\{c_t,k_{t+1},h_{t+1}\}}$ s.t. $c_t + k_{t+1} - (1-\delta)k_t \leq w_t h_t + r_t k_t$

s.t. $h_{t+1} = Bh_{2t}^{\phi}\left(\overline{h_t}\right)^{\psi}$ s.t. $h_t = h_{1t} + h_{2t}$

Firm's problem: $A\widetilde{k}_t^{\alpha}\widetilde{h}_{1t}^{1-\alpha} - w_t\widetilde{h}_{1t} - r_t\widetilde{k}_t \rightarrow \max_{h_t,k_t}$ (2 pt)

Competitive equilibrium: allocations $\{c_t, k_{t+1}, h_{t+1}, h_{1t}, h_{2t}\}$ and $\left\{\widetilde{h}_{1t}, \widetilde{k}_t\right\}$ and $\left\{\overline{h_t}\right\}$ and prices $\{1, r_t, w_t\}$ s.t.

1) allocation 1 solves consumer's problem given prices and the value of $\left\{\overline{h_t}\right\}$

2) allocation 2 solves firm's problem given prices

3) the resource constraints hold: $\widetilde{k}_t = k_t$, $\widetilde{h}_{1t} = h_{1t}$, $c_t+k_{t+1}-(1-\delta)k_t = Ak_t^{\alpha}(h_{1t})^{1-\alpha}$

4) $\overline{h_t} = h_t$ in equilibrium (4 pt)

The difference between the planner's problem and the competitive equilibrium is only in whether to take the derivative w.r.t. $\overline{h_t}$. So we can write the Lagrangian for the planner and then ignore one term:

$$L = \Sigma_{t=0}^{\infty}\beta^t \left(\begin{array}{c} \frac{c_t^{1-\sigma}}{1-\sigma} + \lambda_t\left(Ak_t^{\alpha}(h_t - h_{2t})^{1-\alpha} - c_t - k_{t+1} + (1-\delta)k_t\right) + \\ +\mu_t\left(Bh_{2t}^{\phi}(h_t)^{\psi} - h_{t+1}\right) \end{array} \right) \rightarrow \max_{c_t,k_{t+1},h_{t+1},h_{2t}}$$

FOC: $c_t^{-\sigma} = \lambda_t$

FOC: $\beta\lambda_{t+1}\left(\alpha\frac{y_{t+1}}{k_{t+1}} + 1 - \delta\right) = \lambda_t$

FOC: $\beta\lambda_{t+1}(1-\alpha)\frac{y_{t+1}}{h_{1t+1}} + \boxed{\beta\mu_{t+1}\psi\frac{h_{t+2}}{h_{t+1}}} = \mu_t$

FOC: $\lambda_t(1-\alpha)\frac{y_t}{h_{1t}} = \mu_t\phi\frac{h_{t+1}}{h_{2t}}$

Hence, (5 pt)

$y_t = Ak_t^{\alpha}(h_t - h_{2t})^{1-\alpha}$ $h_{t+1} = Bh_{2t}^{\phi}(h_t)^{\psi}$ $c_t + k_{t+1} - (1-\delta)k_t = y_t$ $h_t = h_{1t} + h_{2t}$

$\beta\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\frac{y_{t+1}}{y_t}\frac{h_{1t}}{h_{1t+1}}\left(\phi + \boxed{\psi\frac{h_{2t+1}}{h_{t+1}}}\right) = \frac{h_{2t}}{h_{t+1}}$ $\beta\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\left(\alpha\frac{y_{t+1}}{k_{t+1}} + 1 - \delta\right) = 1$

Balanced Growth Path:

$\frac{c}{k} + \gamma_k - (1-\delta) = \frac{y}{k}$ $\beta\left(\frac{1}{\gamma_c}\right)^{\sigma}\left(\alpha\frac{y}{k} + 1 - \delta\right) = 1$ $\Rightarrow$ $\gamma_k = \gamma_y = \gamma_c$

$\frac{y_t}{h_{1t}} = A\left(\frac{k_t}{h_{1t}}\right)^{\alpha}$ $\Rightarrow$ $\gamma_{h1} = \gamma_y = \gamma_k$

$h_{t+1} = Bh_{2t}^{\phi}(h_t)^{\psi}$ $\Rightarrow$ $\frac{h_{t+1}}{h_t} = B\frac{h_{2t}^{\phi}}{h_t^{1-\psi}}$ $\Rightarrow$ $\phi * \gamma_{h2} = (1-\psi) * \gamma_h$

$\frac{h_t}{h_{2t}} = \frac{h_{1t}}{h_{2t}} + \frac{h_{2t}}{h_{2t}}$ $\Rightarrow$ $B\frac{h_t^{\psi}}{h_{2t}^{1-\phi}}\frac{h_t}{h_{t+1}} = \frac{h_{1t}}{h_{2t}} + 1$ (5 pt)

For there to be a balanced growth path we need $(1-\phi) * \gamma_{h2} = \psi * \gamma_h$ and $\gamma_{h1} = \gamma_{h2}$

The only way this is possible is if $1 = \psi + \phi$. Another possibility is $\gamma_i = 0$ for all variables which implies a steady-state.

(b) If there is no growth, then $\beta\phi h = h_2$ $h_1 = h - h_2 = (1-\beta\phi)h$ $h^{1-\psi-\phi} = \left[B(\beta\phi)^{\phi}\right]$

$\frac{y}{k} = \frac{\frac{1}{\beta}-1+\delta}{\alpha}$ $\frac{c}{k} = \frac{y}{k} - \delta$ $k = \left(\frac{\alpha A}{\frac{1}{\beta}-1+\delta}\right)^{\frac{1}{1-\alpha}}(1-\beta\phi)h$

Hence, the steady-state is: $h = \left[B(\beta\phi)^{\phi}\right]^{\frac{1}{1-\psi-\phi}}$ $\qquad h_1 = (1-\beta\phi)\,h \qquad h_2 = \beta\phi h$

$$k = \left(\frac{\alpha A}{\frac{1}{\beta}-1+\delta}\right)^{\frac{1}{1-\alpha}}(1-\beta\phi)\,h \qquad c = \left(\frac{\frac{1}{\beta}-1+\delta}{\alpha}-\delta\right)k \qquad \text{(5 pt)}$$

(c) Now assume $1 = \psi + \phi$. We haven't used the equation determining the optimal allocation of human capital yet, which is the only one that depends on whether the externality is taken into account or not. We got that on a balanced growth path all the variables grow at the same rate.

$$\gamma = \frac{h_{t+1}}{h_t} = B\frac{h_{2t}^{\phi}}{h_t^{1-\psi}} = B\frac{h_2^{\phi}}{h^{1-\psi}} = B\left(\frac{h_2}{h}\right)^{\phi} \qquad \Rightarrow \qquad \frac{h_2}{h} = \left(\frac{\gamma}{B}\right)^{\frac{1}{\phi}}$$

$\boxed{\gamma^{\sigma-1} = \beta\phi\left(\frac{B}{\gamma}\right)^{\frac{1}{\phi}}}$ if externality is NOT taken into account (competitive equilibrium).

In a competitive equilibrium the growth rate is: $\gamma = \left(\phi\beta B^{\frac{1}{\phi}}\right)^{\frac{\phi}{\phi(\sigma-1)+1}}$ (5 pt)

(d) $\boxed{\gamma^{\sigma-1} = \beta\left(\phi\left(\frac{B}{\gamma}\right)^{\frac{1}{\phi}}+\psi\right)}$ if externality IS taken into account (planner's solution) (5 pt)

You can't really solve for the planner's growth rate in closed form. However, you can compare the two outcomes. Let's analyze the equation $\frac{1}{\beta}\gamma^{\sigma-1+\frac{1}{\phi}} - \psi\gamma^{\frac{1}{\phi}} - \phi B^{\frac{1}{\phi}} = 0$. Assume $\sigma > 1$ and $0 < \phi < 1$. Call $X = \gamma^{\frac{1}{\phi}}$ which is an increasing function in γ.

Then, the equation is: $F(Z) = Z^{\phi(\sigma-1)+1} - \psi\beta Z - \phi\beta B^{\frac{1}{\phi}} = 0$

If we don't take into account the externality (competitive equilibrium) then the solution is

$$Z^{\phi(\sigma-1)+1} = \phi\beta B^{\frac{1}{\phi}} \qquad Z^* = \left(\phi\beta B^{\frac{1}{\phi}}\right)^{\frac{1}{\phi(\sigma-1)+1}}$$

At this point $F(Z^*) < 0$ and $F'(Z) = \frac{(\phi(\sigma-1)+1)Z^{\phi(\sigma-1)+1}-\psi\beta Z}{Z} > 0$ for Z^*.

Therefore, the solution under the planner's problem has a higher growth rate. That is because in a competitive equilibrium agents over-invest into human capital and under-invest into physical capital, which implies slower growth and lower welfare in equilibrium. (5 pt)