Economics 201A, TA section 6

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1. 2x2 CRS Economy. Basic Facts.

Constant Returns to Scale: $F(\lambda K, \lambda L) = \lambda F(K, L)$
Homotheticity: $MRTS(\lambda K, \lambda L) = \frac{\partial F(\lambda K, \lambda L)}{\partial K} / \frac{\partial F(\lambda K, \lambda L)}{\partial L} = \lambda \frac{\partial F(K,L)}{\partial K} / \lambda \frac{\partial F(K,L)}{\partial L} = MRTS(K, L)$
Marginal cost equal Average cost:
$\min\{r_1 K + r_2 L | F(K, L) \le q\}$ $\quad £ = -r_1 K - r_2 L + \lambda(F(K, L) - q)$
FOC: $r_1 = \lambda \frac{\partial F(K,L)}{\partial K}$ $\quad r_2 = \lambda \frac{\partial F(K,L)}{\partial L}$ $\quad \Rightarrow \quad \frac{\partial F(K,L)}{\partial K} / \frac{\partial F(K,L)}{\partial L} = r_1/r_2$
$C(q, r) = r_1 K + r_2 L = \lambda \left(\frac{\partial F(K,L)}{\partial K} K + \frac{\partial F(K,L)}{\partial L} L \right) = \lambda q$
$MC(q, r) = \frac{\partial C(q,r)}{\partial q} = \lambda = AC(q, r) = \frac{C(q,r)}{q}$
Input demand and cost homogeneous degree one in q
$z(\alpha q, r) = \arg\min \left\{ rz | \frac{F(z)}{\alpha} \ge q \right\} = \alpha \arg\min \left\{ r\frac{z}{\alpha} | F\left(\frac{z}{\alpha}\right) \ge q \right\} = \alpha z(q, r)$
$C(\alpha q, r) = rz(\alpha q, r) = \alpha rz(q, r) = \alpha C(q, r)$
Marginal cost and Average cost constant in q
$MC(q, r) = AC(q, r) = \frac{C(q,r)}{q} = C(1, r) = MC(r) = AC(r)$
Zero profit or infinity, price equal to average cost
$\frac{\pi}{q} = \frac{1}{q}(pq - C(q, r)) = p - C(1, r) = p - AC(r) = 0$
PPF concave. MRTS increasing for A and decreasing for B.

2. 2x2: Cobb-Douglas production (2002 hw3)

There are two outputs and 2 inputs. Technology is Cobb-Douglas

$q_i = K_i^{\alpha_i} L_i^{1-\alpha_i}, \ i \in \{A, B\}$ where $\alpha_A = \frac{1}{5}$ and $\alpha_B = \frac{1}{2}$ $\qquad$ L=K=100

(a) Draw the Edgeworth box for this economy with labor on the horizontal axis. Explain why the Pareto allocations must lie below the diagonal.

(b) If all the resources in the economy are used to produce one good, how much will be produced. Use this answer to indicate the end points of the Production-Possibility-frontier in a neat figure.

(c) Show that if (almost) all inputs are used in the production of commodity B the input price ratio must be 1.

(d) Suppose that $w = r = 1$. Show that the equilibrium price of commodity B is 2 and solve for the equilibrium price of commodity A.

(e) Do the same exercise when (almost) all inputs are used to produce commodity B. (First solve for the equilibrium input price ratio and use this to compute the equilibrium output price ratio.

(f) Hence, or otherwise indicate the range of price ratios for which the country produces both commodities.

a) $F_A(K,L) = K^{1/5}L^{4/5} \qquad F_B(K,L) = K^{1/2}L^{1/2}$

$MRTS_A = \frac{\frac{4}{5L}}{\frac{1}{5K}} = \frac{4K}{L} \qquad MRTS_B = \frac{\frac{1}{2L}}{\frac{1}{2K}} = \frac{K}{L}$

On the diagonal $4 = MRTS_A > MRTS_B = 1$

Above the diagonal $MRTS_A > 4 \qquad MRTS_B < 1$

The only way to have $MRTS_A = MRTS_B$ is going below diagonal.

PE allocations: $\frac{L_A}{4K_A} = \frac{L_B}{K_B}, \qquad K_A + K_B = 100, \qquad L_A + L_B = 100$

$\frac{L_A}{4K_A} = \frac{100-L_A}{100-K_A} \qquad \Rightarrow \qquad K_A = 100\frac{L_A}{400-3L_A},$

$K_B = 100\left(1 - \frac{L_A}{400-3L_A}\right) = 400\frac{100-L_A}{400-3L_A} = 400\frac{L_B}{100+3L_B}$

b) $F_A(K,L)^{\max} = K^{1/5}L^{4/5} = 100 \qquad F_B(K,L)^{\max} = K^{1/2}L^{1/2} = 100$

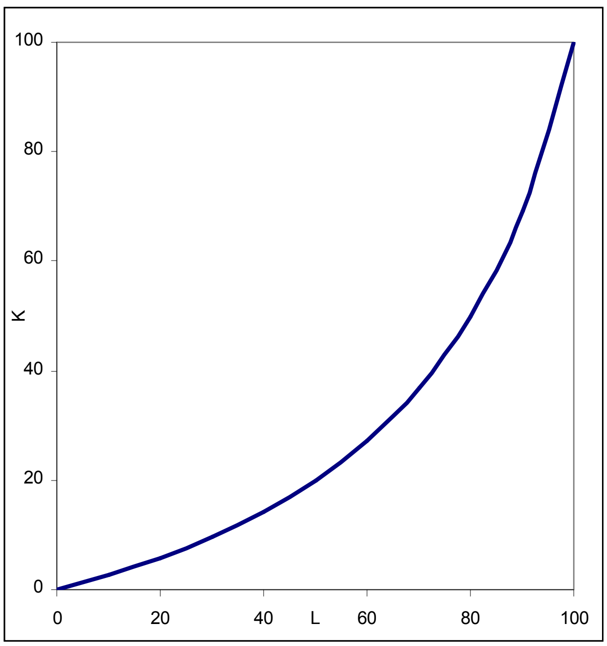
$F_A = \left(100\frac{L_A}{400-3L_A}L_A^4\right)^{1/5} = L_A\sqrt[5]{\frac{100}{400-3L_A}}$

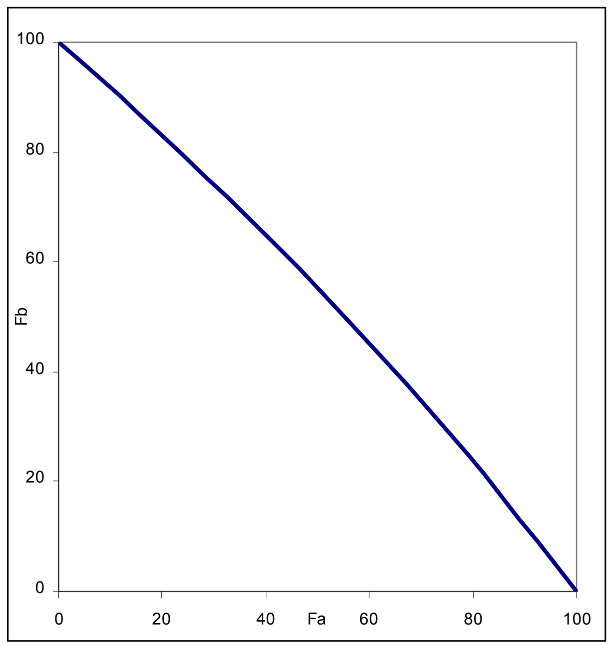
$F_B = \left(400\frac{100-L_A}{400-3L_A}(100-L_A)\right)^{1/2} = 2(100-L_A)\sqrt{\frac{100}{400-3L_A}}$

This defines the PE frontiere in terms of F_A and F_B.

($\max\{F_A | F_B \geq F, K_A + K_B = 100, L_A + L_B = 100\}$)

It is convex, because $MRTS$ is monotonically increasing as L_A goes form 0 to 100.





c) This corresponds to the case when $K_B \to 100$ and $L_B \to 100$.

The equilibrium prices are: $\frac{w}{r} = MRTS_A = MRTS_B = \frac{100}{100} = 1$

d) $C(q,r) = \min\{rK + wL | K^\alpha L^{1-\alpha} \leq q\} \qquad \pounds = -rK - wL + \lambda(K^\alpha L^{1-\alpha} - q)$

FOC: $r = \lambda\frac{\alpha F(K,L)}{K} \qquad w = \lambda(1-\alpha)\frac{F(K,L)}{L} \qquad \Rightarrow \qquad \frac{w}{r} = \frac{1-\alpha}{\alpha}\frac{K}{L}$

$K = \frac{w}{r}\frac{\alpha}{1-\alpha}L \qquad q = K^\alpha L^{1-\alpha} = L\left(\frac{w}{r}\frac{\alpha}{1-\alpha}\right)^\alpha$

$C(q,r) = rK + wL = wL\left(1 + \frac{\alpha}{1-\alpha}\right) = \frac{1}{1-\alpha}wL = \left(\frac{r}{\alpha}\right)^\alpha\left(\frac{w}{1-\alpha}\right)^{1-\alpha}q$

$MC(r) = AC(r) = \left(\frac{r}{\alpha}\right)^\alpha\left(\frac{w}{1-\alpha}\right)^{1-\alpha}$

If $w = r = 1$ then

$p_A = AC_A = (5r)^{1/5}\left(\frac{5w}{4}\right)^{4/5} = (5)^{1/5}\left(\frac{5}{4}\right)^{4/5} = \frac{5}{4^{4/5}}$

$p_B = AC_B = (2r)^{1/2}(2w)^{1/2} = 2$

e) If $K_A \to 100$ and $L_A \to 100$ then $\frac{w}{r} = MRTS_A = 4$
If $w = 4$ and $r = 1$ then $p_A = (5)^{1/5} \left(4 * \frac{5}{4}\right)^{4/5} = 5$
$p_B = (2)^{1/2} (2 * 4)^{1/2} = 4$
f) $p_A \in \left[\frac{5}{4^{4/5}}, 5\right]$ $\quad p_B \in [2, 4]$ $\quad \Rightarrow \quad$ $\frac{p_A}{p_B} = \left[\frac{1}{2}\frac{5}{4^{4/5}}, \frac{5}{4}\right] = [\ 0.825, \quad 1.25\ ]$

3. Welfare Theorems

1st: goods private, local non-satiation ⇒ every WE is PO
2nd: goods private, local non-satiation, quasi-concave ⇒ any PO can be decentralized as WE.
What do we need for that? Money transfers!
1) local non-satiation: realxed if weak PO: cannot make both strictly better off.
2) quasi-concavity: need for existence of WE, when have already chosen PO. Counterexample.
3) goods private: no external effects.

4. 2x2 CES production.

$$q_A = \sqrt{z_{A1} z_{A2}}\,,\quad q_B = (z_{B1}^{1/2} + z_{B2}^{1/2})^2.$$

(a) Solve for the Average Cost functions. (I think the second one is $AC_B = \frac{1}{r_1} + \frac{1}{r_2}$.)

(b) If the equilibrium input prices are $(r_1, r_2) = (1, \frac{1}{4})$ and both commodities are produced, what are the equilibrium output prices.

(c) The aggregate endowment in this economy $\omega = (100, 900)$. What is the range of possible input prices?

(d) Solve also for the range of possible output price ratios.

(e) Depict the production possibility frontier in a neat figure, showing the slopes of the frontier at each corner.

Hint: Start with the maximum input price ratio and set $r_2 = 1$. Then make use of he average cost functions.

(f) In another country the equilibrium input prices are $(r_1, r_2) = (\frac{1}{4}, 1)$. If both commodities are produced, what are the equilibrium output prices?
(g) Could these two countries be trading partners in a world where goods are traded but inputs are not tradable? If not why not? If so, what can you say about the ratio of input endowments in this seconds economy?
(h) In a third economy the input endowment ratio is 1. If the output prices are as in (b) what can you say about production of the two commodities?

a) $F_A(K, L) = z_{1A}^{1/2} z_{2A}^{1/2} \qquad F_B(K, L) = \left(z_{1A}^{1/2} + z_{2A}^{1/2}\right)^2$
$MRTS_{A,1,2} = \frac{z_2}{z_1} \qquad MRTS_{B,1,2} = \left(\frac{z_2}{z_1}\right)^{1/2}$

Cost functions:

A: $C(q,r) = 2q\sqrt{r_1 r_2}$

B: $C(q,r) = q\left(\sum_{i=1}^{n}(r_i)^{1-\sigma}\right)^{\frac{1}{1-\sigma}} = q\left((r_1)^{1-2} + (r_2)^{1-2}\right)^{\frac{1}{1-2}} = \frac{q}{\frac{1}{r_1}+\frac{1}{r_2}}$

b) $r = \left(1, \frac{1}{4}\right) \qquad p_A = 2\sqrt{r_1 r_2} = 1 \qquad p_B = \frac{1}{\frac{1}{r_1}+\frac{1}{r_2}} = \frac{1}{5}$

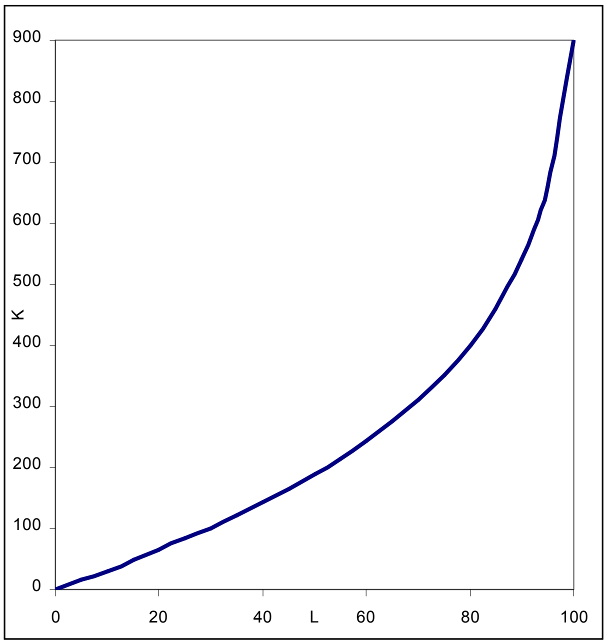
c) $w = (100, 900) \qquad MRTS_A|_w = 9 \qquad MRTS_B|_w = 3$

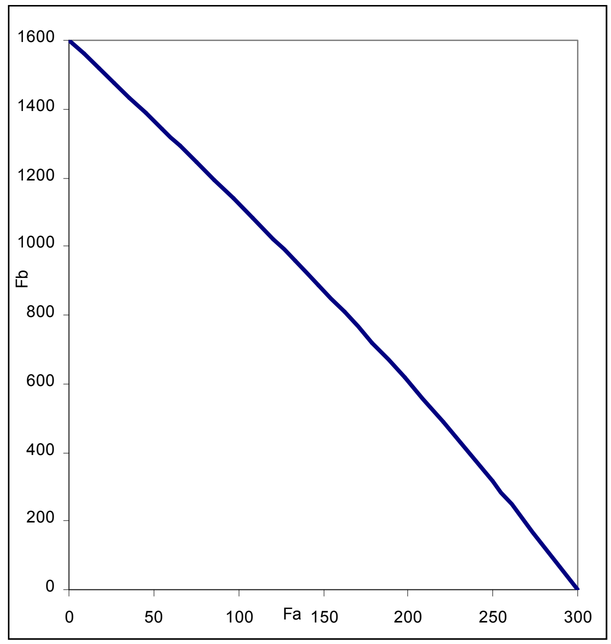
$\Rightarrow \qquad r_1/r_2 \in [3, 9]$

d) $\frac{p_A}{p_B} = 2\sqrt{r_1 r_2}\left(\frac{1}{r_1}+\frac{1}{r_2}\right) = 2\left(\sqrt{\frac{r_2}{r_1}} + \sqrt{\frac{r_1}{r_2}}\right) \in \left[\frac{8\sqrt{3}}{3}, \frac{20}{3}\right]$

e) $F_A^{\max} = 300 \qquad F_B^{\max} = 1600$

$\left(\frac{z_2}{z_1}\right)^2 = \frac{900-z_2}{100-z_1}$, Solution is: $\frac{z_1}{2z_1-200}\left(z_1 \pm \sqrt{z_1^2 - 3600z_1 + 360\,000}\right)$





f) $r = \left(\frac{1}{4}, 1\right) \qquad p_A = 2\sqrt{r_1 r_2} = 1 \qquad p_B = \frac{1}{\frac{1}{r_1}+\frac{1}{r_2}} = \frac{1}{5}$ Same thing.

g) These two countries could always be trading partners.

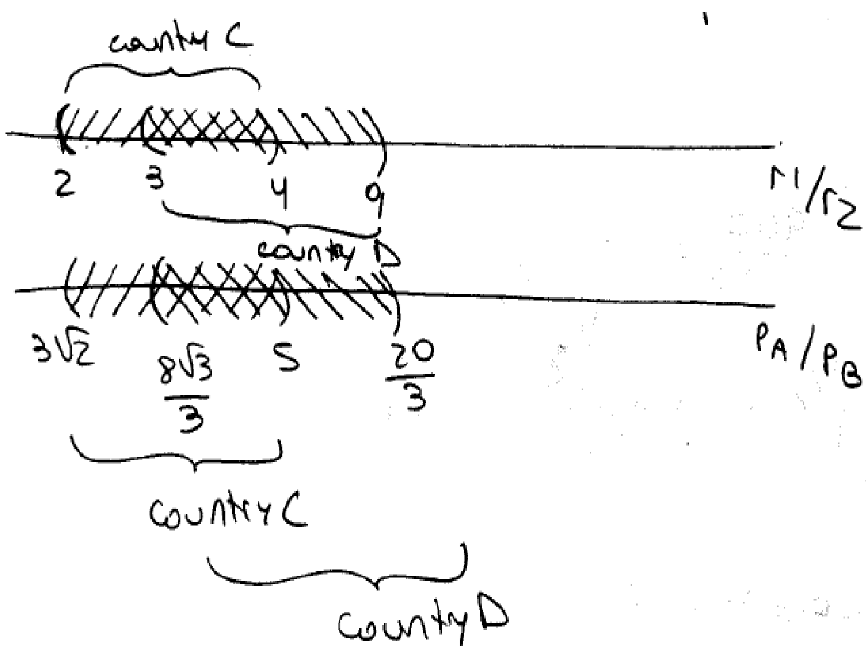
If the other country was relatively richer in labor then the new country would sell good B, and buy good A, and the other way round. There could also be no complete specialization:

For example, if $w = (100, 400) \qquad MRTS_A|_w = 4 \qquad MRTS_B|_w = 2$

$\Rightarrow \qquad r_1/r_2 \in [2, 4] \qquad \frac{p_A}{p_B} = 2\sqrt{r_1 r_2}\left(\frac{1}{r_1}+\frac{1}{r_2}\right) = 2\left(\sqrt{\frac{r_2}{r_1}} + \sqrt{\frac{r_1}{r_2}}\right) \in \left[3\sqrt{2}, 5\right]$

In this case, when $r_1/r_2 \in [3, 4]$ and correspondingly $\frac{p_A}{p_B} \in \left[\frac{8\sqrt{3}}{3}, 5\right]$,

both countries produce both goods, and there is no specialization.



h) If $w_1 = w_2$ then the price vector in the interior is constant: $p_A/p_B = r_1/r_2 = 1$

That means, given the previous price ratios, country (h) will specialize in good A, while the other two countries will produce both goods. That's an illustration of the case, when the difference in relative endowments is so large, that there is complete specializaiton.