Economics 201A, TA section 5 Anton Cheremukhin

October 27, 2006

**Midterm answer key**

1. Choice over time

$U = \ln x_1 + 2\ln x_2 + \ln x_3 + 2\ln x_4 \qquad w = (100, 100, 0, 200)$

a) Preferences are identical and homothetic. Use $U(\lambda x) = U(x) + 6\ln\lambda$.

Homotheticity: $U(x_1) \le U(x_2) \Rightarrow U(x_1) + 6\ln\lambda \le U(x_2) + 6\ln\lambda \Rightarrow U(\lambda x_1) \le U(\lambda x_2)$

Therefore, we can use the representative agent approach

b) We exchange one unit of $x_1$ for half unit of $x_3$ : $\quad x_3 \le \frac{1}{2}(100 - x_1)$

c) $x_2 = w_2 = 100, \quad x_4 = w_4 = 200 \qquad \ln x_1 + \ln x_3 \to \max \quad s.t. \quad x_3 \le \frac{1}{2}(100 - x_1)$

$\ln x_1 + \ln\left[\frac{1}{2}(100 - x_1)\right] \to \max \qquad$ FOC: $\frac{1}{x_1} + \frac{1}{\frac{1}{2}(100-x_1)} * \left(-\frac{1}{2}\right) = 0 \quad \Rightarrow \quad x_1 = 50$

$\Rightarrow \quad x_3 = \frac{1}{2}(100 - 50) = 25 \quad \Rightarrow \quad x = (50, 100, 25, 200)$

d) Prices are determined by the gradient of utilities at point x.

$p = \frac{\partial U/\partial x_i}{\partial U/\partial x_1}|_x = x_1\left(\frac{1}{x_1}, \frac{2}{x_2}, \frac{1}{x_3}, \frac{2}{x_4}\right)|_{(50,100,25,200)} = \left(1, 1, 2, \frac{1}{2}\right)$

e) $p_3 = \frac{p_3^{FS}}{1+r} = \frac{1}{1+r} = 2 \quad \Rightarrow \quad r = -\frac{1}{2}$ Future spot price of good 4: $p_4^{FS} = p_3(1+r) = \frac{1}{4}$

f) NO. The marginal utilities are reflected in the futures prices. The price of good 2 is larger than of good 4. If we could exchange them one for one, we would like to move the goods from tomorrow to today, which is not allowed.

2. Pareto Efficient Allocations

$U_A = \left(10 + x_1^A\right)^{1/2} + \left(x_2^A\right)^{1/2} \qquad U_B = \left(x_1^B\right)^{1/2} + \left(x_2^B\right)^{1/2} \qquad w = (90, 100)$

a) PE: $\max\left\{\left(x_1^B\right)^{1/2} + \left(x_2^B\right)^{1/2} + \lambda\left(\left(10 + 90 - x_1^B\right)^{1/2} + \left(100 - x_2^B\right)^{1/2} - U\right)\right\}$

FOC:$\frac{1}{2}\left(x_1^B\right)^{-1/2} + \lambda\left(-\frac{1}{2}\right)\left(10 + 90 - x_1^B\right)^{-1/2} = 0 \qquad \frac{1}{2}\left(x_2^B\right)^{-1/2} + \lambda\left(-\frac{1}{2}\right)\left(100 - x_2^B\right)^{-1/2} = 0$

$\frac{\left(x_1^B\right)^{-1/2}}{\left(x_2^B\right)^{-1/2}} = \frac{\left(100-x_1^B\right)^{-1/2}}{\left(100-x_2^B\right)^{-1/2}} \quad \Rightarrow \quad \frac{x_1^B}{x_2^B} = \frac{100-x_1^B}{100-x_2^B} = \frac{100}{100} = 1 \quad \Rightarrow \quad x_2^B = x_1^B$

That's the representation of the PE allocations in the interior of the Edgeworth box.

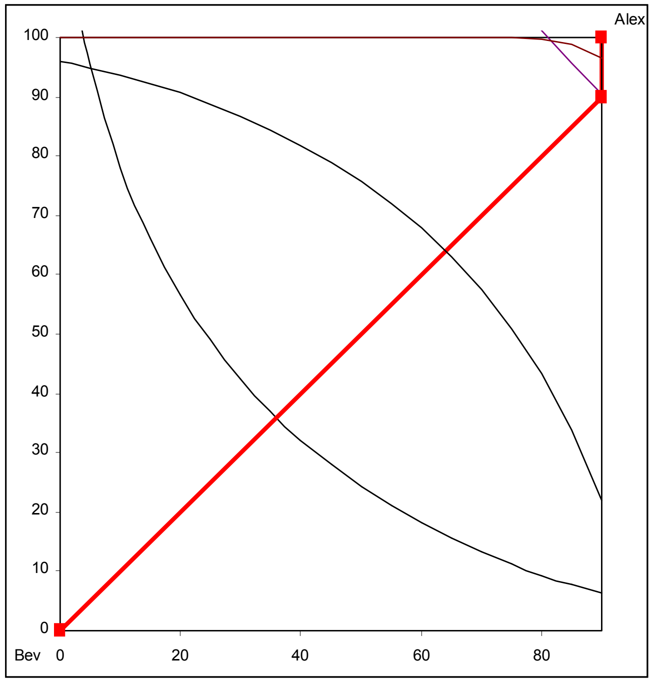
b) The rest of the PE allocations are on the border, basically because the utility functions are concave.

$MRS_A = \frac{\left(10+x_1^A\right)^{-1/2}}{\left(x_2^A\right)^{-1/2}} \qquad MRS_B = \frac{\left(x_1^B\right)^{-1/2}}{\left(x_2^B\right)^{-1/2}}$

Therefore, when $x_2^B < x_1^B \;\Rightarrow\; MRS_A < MRS_B$, so points, improving both, are in the upper-left region between the indifference curves, which is always inside. Similarly, when $x_2^B > x_1^B \;\Rightarrow\; MRS_A > MRS_B$, so points, improving both, are in the lower-right region between the indifference curves. In this case we either hit the interior part of PE, or the vertical segment, described as: $x_1^B = 90, x_2^B \in [90, 100]$. These points are also PE allocations.

c) The prices are determined by the marginal rate of substitution of the agent, who is not in the corner of his budget constraint. When Alex has all the endowment, $\frac{p_1}{p_2} = MRS_A = \frac{(10+w_1)^{-1/2}}{(w_2)^{-1/2}} = 1$

d) When Bev has the entire endowment, $\frac{p_1}{p_2} = MRS_B = \frac{(w_1)^{-1/2}}{(w_2)^{-1/2}} = \sqrt{\frac{10}{9}}$




3. Production and Cost

a) $\min \{r_1 z_1 + r_2 z_2 | z_1 z_2 \leq q\}$ $\qquad \pounds = -r_1 z_1 - r_2 z_2 + \lambda (z_1 z_2 - q)$
FOC: $\quad r_1 = \lambda z_2 \qquad r_2 = \lambda z_1 \qquad \Rightarrow \qquad r_1 z_1 = r_2 z_2$
$q = z_1 z_2 = z_1 z_1 \frac{r_1}{r_2} \qquad \Rightarrow \qquad z_1 = \sqrt{q \frac{r_2}{r_1}}$
$C(q) = r_1 z_1 + r_2 z_2 = 2 r_1 z_1 = 2 r_1 \sqrt{q \frac{r_2}{r_1}} = 2\sqrt{q r_2 r_1}$
b) $q = z_1 (z_2 + z_3)$
In this case $z_2$ and $z_3$ are perfect substitutes, so, only the one that is cheaper will be used. Whatever it is, we have the same problem as before. So, $C(q) = 2\sqrt{q r_1 \min [r_2, r_3]}$.

4. Short and Long Run Responses to an input price change.

Monopoly: $\quad \pi(r) = p(q) q - rz = R(q) - rz$
a) Let $(z_1, q_1)$ be optimal given $r_1$, and $(z_0, q_0)$ be optimal given $r_0$.
Then, $R(q_0) - r_0 z_0 \geq R(q_1) - r_0 z_1 \qquad R(q_1) - r_1 z_1 \geq R(q_0) - r_1 z_0$
Therefore, $R(q_0) - r_0 z_0 + R(q_1) - r_1 z_1 \geq R(q_1) - r_0 z_1 + R(q_0) - r_1 z_0$
This is equivalent to: $\quad (r_1 - r_0)(z_1 - z_0) \leq 0 \qquad \Leftrightarrow \qquad \Sigma \Delta r_j \Delta z_j \leq 0$
If we increase only $r_k$ then $\Sigma \Delta r_j \Delta z_j = \Delta r_k \Delta z_k \leq 0$ and $z_k$ has to fall.
b) Let $(z_1, q_1)$ be optimal given $r_1$, and $(z_0, q_0)$ be optimal given $r_0$.
And $(z^\lambda, q^\lambda)$ be optimal given $r^\lambda = \lambda r_0 + (1-\lambda) r_1$
Then, $R(q_0) - r_0 z_0 \geq R(q^\lambda) - r_0 z^\lambda \qquad \Leftrightarrow \qquad \lambda (R(q_0) - r_0 z_0) \geq \lambda (R(q^\lambda) - r_0 z^\lambda)$
$R(q_1) - r_1 z_1 \geq R(q^\lambda) - r_1 z^\lambda \qquad \Leftrightarrow \qquad (1-\lambda)(R(q_1) - r_1 z_1) \geq (1-\lambda)(R(q^\lambda) - r_1 z^\lambda)$
Therefore, $(1-\lambda)\pi(r_1) + \lambda \pi(r_0) = (1-\lambda)(R(q_1) - r_1 z_1) + \lambda (R(q_0) - r_0 z_0) \geq$
$\geq (1-\lambda)(R(q^\lambda) - r_1 z^\lambda) + \lambda (R(q^\lambda) - r_0 z^\lambda) = R(q^\lambda) - (1-\lambda)(r_1 z^\lambda) - \lambda (r_0 z^\lambda) =$
$= R(q^\lambda) - r^\lambda z^\lambda = \pi(r^\lambda)$
We proved, that $\quad (1-\lambda)\pi(r_1) + \lambda \pi(r_0) \geq \pi(\lambda r_0 + (1-\lambda) r_1)$
That's the definition of a convex function.

c) Profit is a decreasing function of input prices. In the long run we can avoid more constraints than in the short run, so given the same prices the profit must be higher in the long run, or at least not lower. It also has to remain convex.

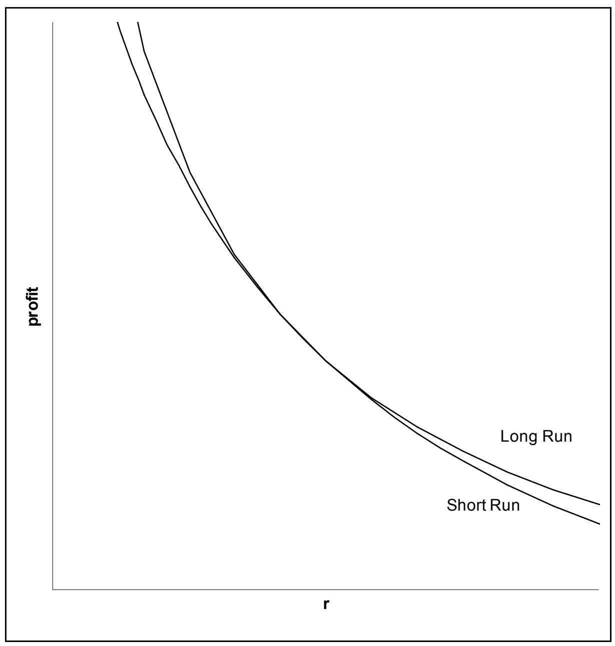
$\pi(r_1) - \pi(r_0) = R(q_0) - r_0 z_0 - R(q_1) + r_1 z_1 \geq R(q_1) - r_0 z_1 - R(q_1) + r_1 z_1$

$\pi(r_1) - \pi(r_0) = R(q_0) - r_0 z_0 - R(q_1) + r_1 z_1 \leq R(q_0) - r_0 z_0 - R(q_0) + r_1 z_0$

Therefore, $(r_1 - r_0) z_1 \leq \pi(r_1) - \pi(r_0) \leq (r_1 - r_0) z_0$

If only one price changes, it follows, that $z_k(r_1) \leq \frac{\pi(r_1)-\pi(r_0)}{r_{1k}-r_{0k}} \leq z_k(r_0)$

This statement is independent of the "run". So in the limit $\frac{\partial \pi(r)}{\partial r_k} = z_k$ is the same in both cases. Since, $\pi^{LR}(r) \geq \pi^{SR}(r)$, $\frac{\partial^2 \pi^{LR}(r)}{\partial r_k^2} \geq \frac{\partial^2 \pi^{SR}(r)}{\partial r_k^2}$. Therefore, "Input price effects on demand are larger in the long run, than in the short run". The graph, representing the Le Chatelier Principle is the following:



**Exercise 1** *F1998, Q2*

*Robinson and Friday live in an island in which coconouts can be produced from labor. The production function is $y = 108\sqrt{L}$. Robinson owns the entire island. He likes coconuts but is unable to work so hires Friday, who has utility function $U_F = x_F - L_F^2$.*

*a) Find PO, WE, and real wage.*

*b) Now assume Robinson can work and his utility is $U_R = x_R - 20L_R$*

a) Robinson consumes all the production minus what Friday consumes.

PO: $\max\left\{U_R|U_F \geq \overline{U}\right\} = \max\left\{108\sqrt{L_F} - x_F|x_F - L_F^2 \geq \overline{U}\right\} = \max\left\{108\sqrt{L_F} - L_F^2 - \overline{U}\right\}$

FOC: $54/\sqrt{L_F} = 2L_F \quad \Rightarrow \quad L_F = 9 \quad \Rightarrow x_F = \overline{U}+81 \quad \Rightarrow x_R = 108\sqrt{L_F} - L_F^2 - \overline{U} = 243 - \overline{U}$

WE: Robinson: $\max\left\{108\sqrt{L_F^D} - \frac{w}{p}L_F^D\right\} \quad \Rightarrow \quad \frac{w}{p} = 54/\sqrt{L_F^D}$ Demand for labor given price.

Friday: $\max\left\{x_F - L_F^2|x_F \leq \frac{w}{p}L_F\right\} \quad \Rightarrow \quad \frac{w}{p} = 2L_F$ Supply of labor given price.

Market clearing: $L_F = L_F^D, \quad x_F + x_R = 108\sqrt{L}$

$\Rightarrow \quad L_F = 9 \quad \frac{w}{p} = 18 \quad \Rightarrow \quad x_F = 162 \quad x_R = 162$

b) PO: $\max\left\{U_R|U_F \geq \overline{U}\right\} = \max\left\{108\sqrt{L_F + L_R} - x_F - 20L_R|x_F - L_F^2 \geq \overline{U}, L_i \geq 0\right\}$

$\pounds = 108\sqrt{L_F + L_R} - x_F - 20L_R + \lambda\left(x_F - L_F^2 - \overline{U}\right)$

FOC: $54/\sqrt{L_F + L_R} = 20 \quad 54/\sqrt{L_F + L_R} = 2\lambda L_F \quad 1 = \lambda \quad x_F = L_F^2 + \overline{U}$

$\Rightarrow \quad L_F = 10 \quad \Rightarrow \quad L_R < 0 \quad \Rightarrow \quad L_R = 0 \quad \Rightarrow L_F = 9$

$\Rightarrow \quad x_F = \overline{U} + 81 \quad \Rightarrow x_R = 108\sqrt{L_F} - L_F^2 - \overline{U} = 243 - \overline{U}$

WE: Robinson: $\max\left\{x_R - 20L_R|x_R \leq \frac{w}{p}L_R + \Pi, L_R \geq 0\right\}$

$\Rightarrow \quad \frac{w}{p} = 20$ for $L_R > 0$ or $L_R = 0, \quad x_R = \frac{w}{p}L_R + \Pi$

Friday: $\max\left\{x_F - L_F^2|x_F \leq \frac{w}{p}L_F\right\} \quad \Rightarrow \quad \frac{w}{p} = 2L_F, \quad x_F = \frac{w}{p}L_F$

Firm: $\Pi = \max\left\{108\sqrt{L} - \frac{w}{p}L\right\} \quad \Rightarrow \quad \frac{w}{p} = 54/\sqrt{L}$

Market clearing: $L_F + L_R = L, \quad x_F + x_R = 108\sqrt{L}$

Combing the equations we get $20 = 2L_F = 54/\sqrt{L_F + L_R}$ which implies $L_F = 10$, $L_R = \left(\frac{54}{20}\right)^2 - 10 = -\frac{271}{100} < 0$

Therefore, here Robinson won't work, and the solution is the same as in (a).