Economics 201A, TA section 4 Anton Cheremukhin

October 20, 2006

1. Homothetic preferences

Definition 1 *$U(x)$ homothetic iff* $U(x^1) \geq U(x^0) \quad \Leftrightarrow \quad U(\theta x^1) \geq U(\theta x^0)$

Preferences are homothetic when "invariant to scaling".

Definition 2 *$U(x)$ homothetic iff* $\forall x \forall \lambda > 0 \exists \mu > 0 : \nabla U(\lambda x) = \mu \nabla U(x)$

Preferences are homothetic when "along a ray the slope of indifference curves is constant".

Proposition 1 *$U(x)$ homothetic* $\Rightarrow x(p, \theta I) = \theta x(p, I)$

Proof. 1) $x(p, I)$ is feasible given I: $px(p, I) = I$
therefore, $\theta x(p, I)$ is feasible given θI : $\theta px(p, I) = \theta I$
2) $x(p, \theta I)$ is feasible given θI: $px(p, \theta I) = \theta I$
therefore, $\frac{1}{\theta} x(p, \theta I)$ is feasible given I : $\frac{1}{\theta} px(p, \theta I) = I$
3) $x(p, \theta I)$ is optimal given θI: $U(\theta x(p, I)) \leq U(x(p, \theta I))$
4) $x(p, I)$ is optimal given I: $U\left(\frac{1}{\theta} x(p, \theta I)\right) \leq U(x(p, I))$
5) $U(x(p, I)) \geq U\left(\frac{1}{\theta} x(p, \theta I)\right) \quad \Rightarrow \quad U(\theta x(p, I)) \geq U(x(p, \theta I))$
Combining 3 and 5, we see that $U(\theta x(p, I)) = U(x(p, \theta I))$.

If U is strictly increasing and strictly quasiconcave, the consumer's problem always has a unique solution, and, hence, $x(p, \theta I) = \theta x(p, I)$. ■

2. Law of demand

Let x_1^c be cost minimizing given p_1 and x_2^c be cost minimizing given p_2.
Then, $p_1 x_1^c \leq p_1 x_2^c$ and $p_2 x_2^c \leq p_2 x_1^c$. Taking the sum of them we get:
$p_1 x_1^c - p_2 x_1^c \leq p_1 x_2^c - p_2 x_2^c \quad \Leftrightarrow \quad (p_1 - p_2)(x_1^c - x_2^c) \leq 0 \quad \Leftrightarrow \quad \Delta p_i \Delta x_i^c \leq 0$

3. Law of supply

Let y_1 be profit maximizing given p_1 and x_2 be profit maximizing given p_2.
Then, $p_1 y_1 \geq p_1 y_2$ and $p_2 y_2 \geq p_2 y_1$. Taking the sum of them we get:
$p_1 y_1 - p_2 y_1 \geq p_1 y_2 - p_2 y_2 \quad \Leftrightarrow \quad (p_1 - p_2)(y_1 - y_2) \geq 0 \quad \Leftrightarrow \quad \Delta p_j \Delta y_j \geq 0$

Corollary 1 *The profit function is convex.*
$\Pi(\lambda p_1 + (1-\lambda) p_2) = \max_y \{\lambda p_1 y + (1-\lambda) p_2 y\} \leq \max_y \{\lambda p_1 y\} + \max_y \{(1-\lambda) p_2 y\} =$
$= \lambda \Pi(p_1) + (1-\lambda) \Pi(p_2)$

4. **Quasiconcavity in problem 2**

Proof. $q^{\frac{\sigma-1}{\sigma}} = (a_1 z_1)^{\frac{\sigma-1}{\sigma}} + (a_2 z_2)^{\frac{\sigma-1}{\sigma}} \qquad z_2 = \frac{1}{a_2}\left(q^{\frac{\sigma-1}{\sigma}} - (a_1 z_1)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$

Think of $0 < z_1 < q/a_1$ when $\sigma > 1$ and $z_1 > q/a_1$ when $0 < \sigma < 1$.

$\frac{\partial z_2}{\partial z_1} = -\frac{a_1}{a_2 (a_1 z_1)^{\frac{1}{\sigma}}}\left(q^{\frac{\sigma-1}{\sigma}} - (a_1 z_1)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} = -\frac{a_1 (a_2 z_2)^{\frac{1}{\sigma}}}{a_2 (a_1 z_1)^{\frac{1}{\sigma}}} < 0$

$\frac{\partial^2 z_2}{\partial z_1^2} = \frac{q}{\sigma}\frac{a_1}{a_2 z_1}\frac{1}{q^{\frac{1}{\sigma}}(a_1 z_1)^{\frac{1}{\sigma}}}\left(q^{\frac{\sigma-1}{\sigma}} - (a_1 z_1)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{2-\sigma}{\sigma-1}} = \frac{1}{\sigma}\frac{a_1}{a_2}\frac{1}{z_1}\frac{q^{\frac{\sigma-1}{\sigma}}}{(a_2 z_2)^{\frac{\sigma-1}{\sigma}}}\frac{(a_2 z_2)^{\frac{1}{\sigma}}}{(a_1 z_1)^{\frac{1}{\sigma}}} > 0$

So, the upper-contoursets are strictly convex, and the production function is strictly quasi-concave. ■

Excess demand: $e(p) = \Sigma x_i(p) - \Sigma y_j(p) - \Sigma w_k$

Market clearing condition: either $e(p) = 0$ or $p = 0$ and $e(p) \leq 0$.

Walrasian Equilibrium: {allocation, price vector (≥ 0)} s.t.

1) allocations are the optimal solutions of agents' problems given prices

2) prices clear the market.

Pareto Optimal: $\{x_i\}_{i=1}^n$, s.t. $\nexists \{x_i'\}_{i=1}^n : \forall i, u(x_i') \geq u(x_i), \quad \exists i_0, u\left(x_{i_0}'\right) > u(x_{i_0})$

"nobody is worse off, and smb is strictly better off".

First Welfare Theorem

If preferences exhibit local non-satiation and goods are private, then any WE is PO.

Second Welfare Theorem

If preferences are also strictly increasing and quasi-concave, then any PO can be decentralized as a WE.

Can you find an example when FWT holds but SWT doesn't...?

Identical homothetic preferences

For agents with identical homothetic preferences, aggregate excess demand can be derived from maximization of a utility function of a representative agent whose endowment is the sum of individual endowments. Such an economy has unique equilibrium, in which each agent consumes a fraction of the aggregate endowment. The equilibrium prices can be found from: $\left.\frac{\partial U(x)/\partial x_i}{\partial U(x)/\partial x_j}\right|_{x=w} = \frac{p_i}{p_j}$

Exercise 1 *F1998, Q2*

Robinson and Friday live in an island in which coconouts can be produced from labor. The production function is $y = 108\sqrt{L}$. *Robinson owns the entire island. He likes coconuts but is unable to work so hires Friday, who has utility function* $U_F = x_F - L_F^2$.

a) Find PO, WE, and real wage.

b) Now assume Robinson can work and his utility is $U_R = x_R - 20L_R$

a) Robinson consumes all the production minus what Friday consumes.

PO: $\max\left\{U_R | U_F \geq \overline{U}\right\} = \max\left\{108\sqrt{L_F} - x_F | x_F - L_F^2 \geq \overline{U}\right\} = \max\left\{108\sqrt{L_F} - L_F^2 - \overline{U}\right\}$

FOC: $54/\sqrt{L_F} = 2L_F \quad \Rightarrow \quad L_F = 9 \quad \Rightarrow x_F = \overline{U} + 81 \quad \Rightarrow x_R = 108\sqrt{L_F} - L_F^2 - \overline{U} = 243 - \overline{U}$

WE: Robinson: $\max\left\{108\sqrt{L_F^D} - \frac{w}{p}L_F^D\right\} \quad \Rightarrow \quad \frac{w}{p} = 54/\sqrt{L_F^D}$ Demand for labor given price.

Friday: $\max\left\{x_F - L_F^2 | x_F \leq \frac{w}{p}L_F\right\} \quad \Rightarrow \quad \frac{w}{p} = 2L_F$ Supply of labor given price.

Market clearing: $L_F = L_F^D, \qquad x_F + x_R = 108\sqrt{L}$

$\Rightarrow \qquad L_F = 9 \qquad \frac{w}{p} = 18 \qquad \Rightarrow \qquad x_F = 162 \qquad x_R = 162$

b) PO: $\max\left\{U_R|U_F \geq \overline{U}\right\} = \max\left\{108\sqrt{L_F+L_R} - x_F - 20L_R | x_F - L_F^2 \geq \overline{U}, L_i \geq 0\right\}$

$\pounds = 108\sqrt{L_F+L_R} - x_F - 20L_R + \lambda\left(x_F - L_F^2 - \overline{U}\right)$

FOC: $54/\sqrt{L_F+L_R} = 20 \qquad 54/\sqrt{L_F+L_R} = 2\lambda L_F \qquad 1 = \lambda \qquad x_F = L_F^2 + \overline{U}$

$\Rightarrow \qquad L_F = 10 \qquad \Rightarrow \qquad L_R < 0 \qquad \Rightarrow \qquad L_R = 0 \qquad \Rightarrow L_F = 9$

$\Rightarrow \qquad x_F = \overline{U} + 81 \qquad \Rightarrow x_R = 108\sqrt{L_F} - L_F^2 - \overline{U} = 243 - \overline{U}$

WE: Robinson: $\max\left\{x_R - 20L_R | x_R \leq \frac{w}{p}L_R + \Pi, L_R \geq 0\right\}$

$\Rightarrow \qquad \frac{w}{p} = 20$ for $L_R > 0$ or $L_R = 0, \qquad x_R = \frac{w}{p}L_R + \Pi$

Friday: $\max\left\{x_F - L_F^2 | x_F \leq \frac{w}{p}L_F\right\} \qquad \Rightarrow \qquad \frac{w}{p} = 2L_F, \qquad x_F = \frac{w}{p}L_F$

Firm: $\Pi = \max\left\{108\sqrt{L} - \frac{w}{p}L\right\} \qquad \Rightarrow \qquad \frac{w}{p} = 54/\sqrt{L}$

Market clearing: $L_F + L_R = L, \qquad x_F + x_R = 108\sqrt{L}$

Combing the equations we get $20 = 2L_F = 54/\sqrt{L_F+L_R}$ which implies $L_F = 10$, $L_R = \left(\frac{54}{20}\right)^2 - 10 = -\frac{271}{100} < 0$

Therefore, here Robinson won't work, and the solution is the same as in (a).