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Comparing prospects

Consider the prospects $\tilde{c}_A = (c, \pi_A)$ and $\tilde{c}_B = (c, \pi_B)$ where $c = (a, a+\delta, a+2\delta)$, $\delta > 0$.

The expected utility of a prospect is

$$U(c,\pi) = \pi_1 v_1 + \pi_2 v_2 + \pi_3 v_3 \text{ where } v_s \equiv v(c_s)$$

Since probabilities sum to 1,

$$U(c,\pi) = \pi_1 v_1 + (1-\pi_1-\pi_3)v_2 + \pi_3 v_3$$

$$= v_2 + \pi_3(v_3 - v_2) - \pi_1(v_2 - v_1) \qquad (1.1)$$

Result 1: Suppose that (i) $\pi_{A1} \le \pi_{B1}$ and (ii) $\pi_{A1} + \pi_{A2} \le \pi_{B1} + \pi_{B2}$ (equivalently $\pi_{A3} \ge \pi_{B3}$). If $v(\cdot)$ is increasing, $U(c,\pi_A) \ge U(c,\pi_B)$ and if $v(\cdot)$ is decreasing $U(c,\pi_A) \le U(c,\pi_B)$.

Conditions (i) and (ii) can be expressed as follows.

$$\Pr\{\tilde{c}_A \le c_1\} \le \Pr\{\tilde{c}_B \le c_1\} \text{ and } \Pr\{\tilde{c}_A \le c_2\} < \Pr\{\tilde{c}_B \le c_2\}.$$

Thus if utility is increasing, a prospect is strictly preferred if there is less probability mass in every left tail. Note, in particular, that if this is the case, the expected value of $\tilde{c}_A$ is greater than the expected value $\tilde{c}_B$.

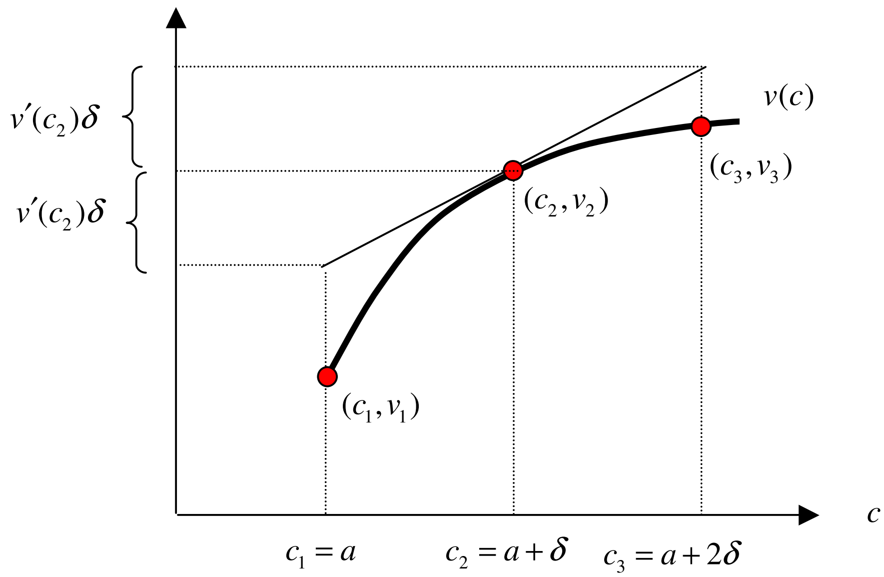
Consider the special linear case $v(c) = c$, from (1.1),

$$U(c,\pi) = E\{\tilde{c}\} = c_2 + (\pi_3 - \pi_1)\delta$$

Result 2: $E\{\tilde{c}_A\} = E\{\tilde{c}_B\}$ if and only if $\pi_{A1} - \pi_{A3} = \pi_{B1} - \pi_{B3}$.

Suppose that the two alternatives have the same mean return and $\pi_{B1} = \pi_{A1} + \Delta$, $\Delta > 0$. From Result 2, it follows that $\pi_{B3} = \pi_{A3} + \Delta$. Also, since probabilities sum to 1, $\pi_{B2} = \pi_{A2} - 2\Delta$. Note that $\tilde{c}_B$ has more probability mass in both tails than $\tilde{c}_A$. Thus we may describe the distribution of $\tilde{c}_B$ as a mean preserving spread of the distribution of $\tilde{c}_A$.

Thus far we have not exploited the curvature of $v(\cdot)$. Suppose, as depicted below, that $v(\cdot)$ is strictly concave.



Since the tangent line lies above the curve,

$$v_2 - v_1 > v'(c_2)\delta > v_3 - v_2$$

From (1.1),

$$U(c,\pi) = v_2 + \pi_3(v_3 - v_2) - \pi_1(v_2 - v_1)$$

Hence

$$U(c,\pi_A) - U(c,\pi_B) = (v_3 - v_2)(\pi_{A3} - \pi_{B3}) - (v_2 - v_1)(\pi_{A1} - \pi_{B1}).$$

If $v(c)$ is strictly concave, $v_3 - v_2 < v_2 - v_1$. Then since $\pi_{A3} < \pi_{B3}$,

$$U(c,\pi_A) - U(c,\pi_B) > (v_2 - v_1)(\pi_{A3} - \pi_{B3}) - (v_2 - v_1)(\pi_{A1} - \pi_{B1})$$

$$= (v_2 - v_1)[(\pi_{A3} - \pi_{B3}) - (\pi_{A1} - \pi_{B1})].$$

By Result 2, the bracketed expression is zero if the two distributions have the same mean. We have therefore proved that a mean preserving spread of the distribution lowers $E\{v(\tilde{c})\}$ if v is concave. If v is convex the inequality is reversed.

Result 3: If $\tilde{c}_B$ is a mean preserving spread of $\tilde{c}_A$ then $E\{v(\tilde{c}_A)\} \geq E\{v(\tilde{c}_B)\}$ if $v(\cdot)$ is concave and $E\{v(\tilde{c}_A)\} \leq E\{v(\tilde{c}_B)\}$ if $v(\cdot)$ is convex.

Continuous Distributions

Suppose that both $\tilde{c}_A$ and $\tilde{c}_B$ both have supports (possibly different) within the interval $[\alpha, \beta]$ and probability density functions $f_A(c)$ and $f_B(c)$. We write the c.d.f.s as $F_A(c)$ and $F_B(c)$ and the integrals of the of the c.d.f.s as $T_A(c)$ and $T_B(c)$.

Definition: First Order Stochastic Dominance

$\tilde{c}_A$ exhibits FOSD over $\tilde{c}_B$ if $\Pr\{\tilde{c}_A \le c\} \equiv F_A(c) \le F_B(c) \equiv \Pr\{\tilde{c}_B \le c\},\ c \in [\alpha, \beta]$

In words, $\tilde{c}_A$ dominates $\tilde{c}_B$ if, for all *c*, the probability mass in the tail to the left of *c* is smaller for $\tilde{c}_A$ than it is for $\tilde{c}_B$.

If $v(\cdot)$ is an increasing function it is intuitively clear that $E\{v(\tilde{c}_A)\} \ge E\{v(\tilde{c}_B)\}$ since the latter has higher weight on bad outcomes. We state and derive this result below.

Proposition 1: Suppose that $\tilde{c}_A$ exhibits FOSD over $\tilde{c}_B$ .. If *v* is increasing then $E\{v(\tilde{c}_A)\} - E\{v(\tilde{c}_B)\} \ge 0$ and if *v* is decreasing then $E\{v(\tilde{c}_A)\} - E\{v(\tilde{c}_B)\} \le 0$.

Proof: $E\{v(\tilde{c}_A)\} - E\{v(\tilde{c}_B)\} = \int_\alpha^\beta v(c) f_A(c)dc - \int_\alpha^\beta v(c) f_B(c)dc$

Integrating by parts and noting that $F_A(\alpha) = F_B(\alpha) = 0$ and $F_A(\beta) = F_B(\beta) = 1$,

$$E\{v(\tilde{c}_A)\} - E\{v(\tilde{c}_B)\} = \int_\alpha^\beta -v'(c)F_A(c)dc - \int_\alpha^\beta -v'(c)F_B(c)dc$$

$$= \int_\alpha^\beta v'(c)[F_B(c) - F_A(c)]dc\,. \qquad (1.1)$$

Thus, if $\tilde{c}_A$ exhibits FOSD over $\tilde{c}_B$, $E\{v(\tilde{c}_A)\} - E\{v(\tilde{c}_B)\} \ge 0$ if *v* is increasing and the inequality is reversed if *v* is decreasing.

QED

This is the generalization of Result 1. We now seek to generalize Results 2 and 3.
First consider the difference in means. From (1.1),

$$E\{\tilde{c}_A\} - E\{\tilde{c}_B\} = \int_\alpha^\beta -F_A(c)dc - \int_\alpha^\beta -F_B(c)dc$$

$$= \int_\alpha^\beta [F_B(c) - F_A(c)]dc = T_A(\beta) - T_B(\beta) .$$

Hence we have the following proposition.

Proposition 2: The distributions $\tilde{c}_A$ and $\tilde{c}_B$ have the same mean if the area under the two c.d.f.s is the same, that is $\int_\alpha^\beta F_A(c)dc = \int_\alpha^\beta F_B(c)dc$.

Consider the first of the two figures below. If the two shaded areas are equal then the areas under the two p.d.f.s are equal. Note that the slope of $F_A(c)$ is increasing to the left of γ while the slope of $F_B(c)$ is decreasing. That is, the density of $\tilde{c}_A$ is increasing and the density of $\tilde{c}_B$ is decreasing. The opposite is true to the right of γ. The probability density functions must therefore be as depicted. Note also that for any $c < \beta$,

$$T_A(c) = \int_\alpha^c F_A(x)dx < \int_\alpha^c F_B(x)dx = T_B(c)$$

Since the two distributions have the same means it is natural to call $\tilde{c}_B$, with its greater weight in the tails, a mean preserving spread of $\tilde{c}_A$.

Definition: Mean Preserving Spread

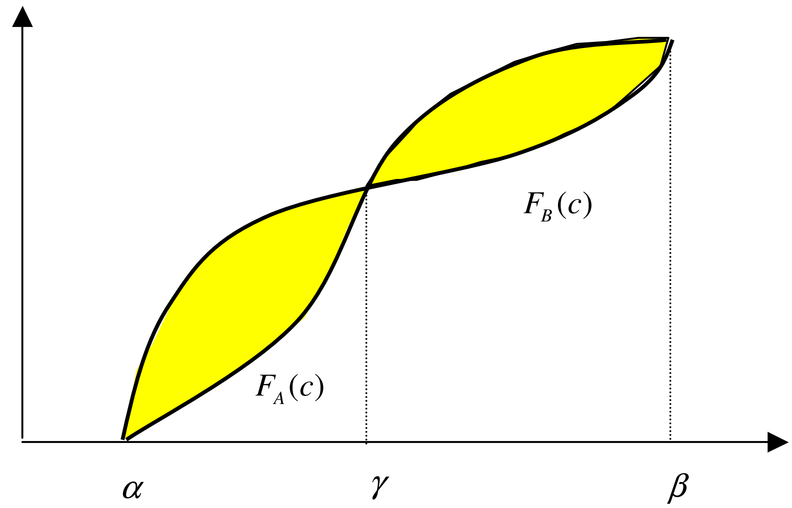
$\tilde{c}_B$ is a mean preserving spread of $\tilde{c}_A$ if for all $c \in [\alpha, \beta]$

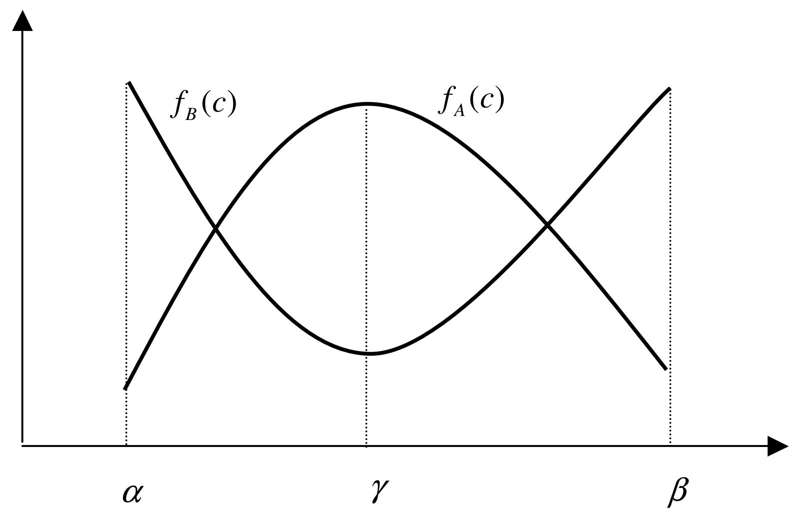
$$T_A(c) = \int_\alpha^c F_A(x)dx \le \int_\alpha^c F_B(x)dx = T_B(c) \text{ and } T_A(\beta) = T_B(\beta) .$$

From (1.1),

$$E\{v(\tilde{c}_A)\} - E\{v(\tilde{c}_B)\} = \int_\alpha^\beta v'(c)[F_B(c) - F_A(c)]dc .$$

Integrating again by parts,





$$E\{v(\tilde{c}_A)\} - E\{v(\tilde{c}_B)\} = v'(c)[T_B(c) - T_A(c)]\Big|_{\alpha}^{\beta} - \int_{\alpha}^{\beta} v''(c)[T_B(c) - T_A(c)]dc$$

Appealing to the definition of a mean preserving spread, we have the following proposition.

Proposition 3: If $\tilde{c}_B$ is a mean preserving spread of $\tilde{c}_A$ then $E\{v(\tilde{c}_A)\} \geq E\{v(\tilde{c}_B)\}$ if $v(\cdot)$ is concave and $E\{v(\tilde{c}_A)\} \leq E\{v(\tilde{c}_B)\}$ if $v(\cdot)$ is convex.

Application: The savings decision

An individual has current income y_1 and uncertain future income $\tilde{y}_2$. The interest rate is *r*. Thus if she saves *S* her consumption bundle is $(c_1, \tilde{c}_2) = (y_1 - S, \tilde{y}_2 + (1+r)S)$. She has a separable lifetime utility function thus her expected lifetime utility is

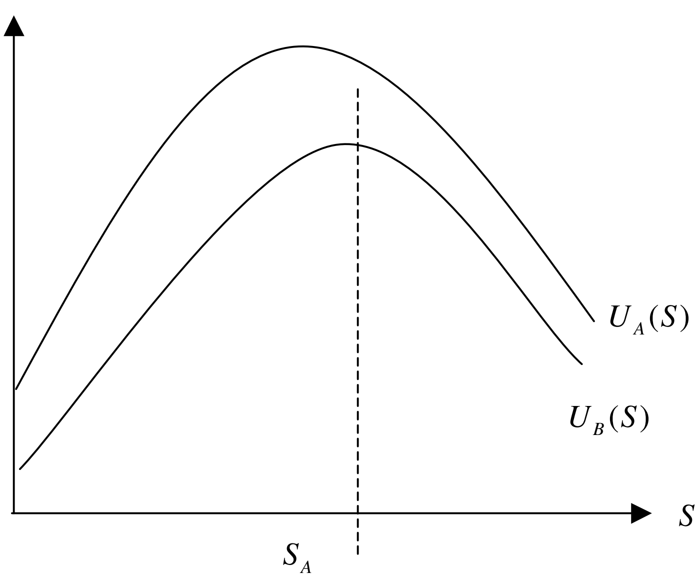
$$U(S) = v_1(y_1 - S) + E\delta v_2(\tilde{y}_2 + (1+r)S)\,. \qquad (1.2)$$

We assume that both $v_1(\cdot)$ and $v_2(\cdot)$ are concave. It is left as an exercise to confirm that $U(S)$ is a concave function. We also assume that the individual exhibits decreasing absolute aversion to risk. Hence

$$\frac{d}{dc}\left(-\frac{v_2''(c)}{v'(c)}\right) = -\frac{v_2'''v_2' - (v_2'')^2}{(v_2')^2} < 0\,.$$

Note that a necessary condition for this inequality to hold is $v_2'''(c_2) > 0$. **Thus the function $v_2'(c)$ is decreasing and convex.**

Fix the level of saving and ask what happens to her expected utility if the distribution of second period income changes from $\tilde{y}_{2A}$ to $\tilde{y}_{2B}$ and the former first order stochastically dominates the latter. By Proposition1, the consumer is worse off. Since this is true for all *S*, the two expected lifetime utility curves must be as depicted.



As shown the optimal savings under $\tilde{y}_{2A}$ is S_A. Note that the picture is drawn so that the slope of $U_B(S)$ is strictly positive at S_A. Thus under the distribution $\tilde{y}_{2B}$, she saves more.

To see that this will be the case, differentiate (1.2).

$$U_B'(S) = -v_1'(y_1 - S) + \delta(1+r)Ev_2'(\tilde{y}_{2B} + (1+r)S)$$

Since $v_2'(c)$ is a decreasing function, it follows from Proposition 1 that

$E\{v_2'(\tilde{c}_B)\} \geq E\{v_2'(\tilde{c}_A)\}$. Hence $U_B'(S) \geq U_A'(S)$ and so $U_B'(S_A) \geq U_A'(S_A)$.

The intuition for this result is clearest when all the realizations under $\tilde{c}_B$ are less than all the possible realizations under $\tilde{c}_A$. This is an extreme case of First Order Stochastic Dominance. If second period income is lower with probability 1 then the consumer will save more to offset the decline in second period income.

Finally suppose that the new income distribution is a mean preserving spread of the old distribution. We know that $v_2'(c)$ is a convex function. Thus, appealing to Proposition 3, expected marginal utility is higher under $\tilde{y}_{2B}$. Then again $U_B'(S) \geq U_A'(S)$ and so the response is to increase saving.