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1. **A Scaled Jump-Diffusion**

The return equation for the scaled discrete-time jump diffusion is:

$R_{t+h} = 1 + (\mu - r)\,h + \sigma_B \varepsilon_{t+h} + \sigma_J v_{t+h}$,

where $\varepsilon_{t+h} = \begin{cases} -\sqrt{h} & \text{with probability } \frac{1}{2} \\ +\sqrt{h} & \text{with probability } \frac{1}{2} \end{cases}$ $\quad v_{t+h} = \begin{cases} -1 & \text{with probability } \frac{\lambda h}{2} \\ 0 & \text{with probability } 1-\lambda h \\ +1 & \text{with probability } \frac{\lambda h}{2} \end{cases}$

The diffusion component and the jump component are mutually independent, hence, the return process is IID and has the following structure:

$$R_{t+h} = \begin{cases} 1 + (\mu - r)\,h - \sigma_J & \text{with probability } \frac{\lambda h}{2} = p_{DJ} \\ 1 + (\mu - r)\,h - \sigma_B\sqrt{h} & \text{with probability } \frac{1-\lambda h}{2} = p_D \\ 1 + (\mu - r)\,h + \sigma_B\sqrt{h} & \text{with probability } \frac{1-\lambda h}{2} = p_U \\ 1 + (\mu - r)\,h + \sigma_J & \text{with probability } \frac{\lambda h}{2} = p_{UJ} \end{cases}$$

For the martingale measure to be unique we need four assets including the risk-free asset corresponding to the four branches in each node of the tree. Therefore, we define $R^n_{t+h} = 1 + (\mu^n - r)\,h + \sigma^n_B \varepsilon_{t+h} + \sigma^n_J v_{t+h}$, $n = 1,2,3$ the three risky assets, and $R^f_{t+h} = 1$ for the risk-free asset. To compute the equivalent martingale measure we evaluate the fundamental equation of asset pricing and apply it to the four assets: $\widetilde{E}\left[R^n_{t+h}|\mathcal{F}_t\right] = 1$.

$1 + (\mu^n - r)\,h - \widetilde{p}_{DJ}\sigma^n_{DJ} - \widetilde{p}_D\sigma^n_{DB}\sqrt{h} + \widetilde{p}_U\sigma^n_{UB}\sqrt{h} + \widetilde{p}_{UJ}\sigma^n_{UJ} = 1 \qquad n = 1,2,3$

$\widetilde{p}_{DJ} + \widetilde{p}_D + \widetilde{p}_U + \widetilde{p}_{UJ} = 1$

This gives us a system of 4 equations in 4 unknowns:

$$\begin{bmatrix} -\sigma^1_{DJ} & -\sigma^1_{DB}\sqrt{h} & \sigma^1_{UB}\sqrt{h} & \sigma^1_{UJ} \\ -\sigma^2_{DJ} & -\sigma^2_{DB}\sqrt{h} & \sigma^2_{UB}\sqrt{h} & \sigma^2_{UJ} \\ -\sigma^3_{DJ} & -\sigma^3_{DB}\sqrt{h} & \sigma^3_{UB}\sqrt{h} & \sigma^3_{UJ} \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \widetilde{p}_{DJ} \\ \widetilde{p}_D \\ \widetilde{p}_U \\ \widetilde{p}_{UJ} \end{bmatrix} = \begin{bmatrix} (r - \mu^1)\,h \\ (r - \mu^2)\,h \\ (r - \mu^3)\,h \\ 1 \end{bmatrix}$$

The solution $\left[\ \widetilde{p}_{DJ} \ \ \widetilde{p}_D \ \ \widetilde{p}_U \ \ \widetilde{p}_{UJ}\ \right]$ is the equivalent martingale measure.

We can make any assumptions about $\sigma^n_{\frac{U}{D}J}$ and σ^n_B as long as the matrix on the LHS is invertible and the solution has only positive elements. Then the stochastic discount factor will be:

$$X_{t+h} = \frac{Z_{t+h}}{Z_t} = \begin{cases} 2\frac{\widetilde{p}_{DJ}}{\lambda h} & \text{if the state is DJ} \\ 2\frac{\widetilde{p}_D}{1-\lambda h} & \text{if the state is D} \\ 2\frac{\widetilde{p}_U}{1-\lambda h} & \text{if the state is U} \\ 2\frac{\widetilde{p}_{UJ}}{\lambda h} & \text{if the state is UJ} \end{cases}$$

If we assume that both $\sigma^n_{DJ} = \sigma^n_{UJ}$ and $\sigma^n_{UB} = \sigma^n_{DB}$ then the matrix on the LHS is non-invertible:

$$\begin{bmatrix} -\sigma^1_J & -\sigma^1_B\sqrt{h} & \sigma^1_B\sqrt{h} & \sigma^1_J \\ -\sigma^2_J & -\sigma^2_B\sqrt{h} & \sigma^2_B\sqrt{h} & \sigma^2_J \\ -\sigma^3_J & -\sigma^3_B\sqrt{h} & \sigma^3_B\sqrt{h} & \sigma^3_J \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \widetilde{p}_{DJ} \\ \widetilde{p}_D \\ \widetilde{p}_U \\ \widetilde{p}_{UJ} \end{bmatrix} = \begin{bmatrix} (r - \mu^1)\,h \\ (r - \mu^2)\,h \\ (r - \mu^3)\,h \\ 1 \end{bmatrix}$$

Therefore, we need to allow at least the jumps up and down for the assets to be of different size.

Let's try a case when one of the assets is a pure brownian motion, and the other two assets also have a brownian-motion component, but one jumps down only, and the other jumps up only.

Define: $\begin{bmatrix} 0 & -\sigma_B^1\sqrt{h} & \sigma_B^1\sqrt{h} & 0 \\ -\sigma_{DJ} & -\sigma_B^2\sqrt{h} & \sigma_B^2\sqrt{h} & 0 \\ 0 & -\sigma_B^3\sqrt{h} & \sigma_B^3\sqrt{h} & \sigma_{UJ} \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -e & e & 0 \\ -a & -b & b & 0 \\ 0 & -c & c & d \\ 1 & 1 & 1 & 1 \end{bmatrix}$

Then, $\begin{bmatrix} 0 & -e & e & 0 \\ -a & -b & b & 0 \\ 0 & -c & c & d \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{b}{a}\frac{1}{e} & -\frac{1}{a} & 0 & 0 \\ -\frac{ad-ac+bd}{ad}\frac{1}{2e} & \frac{1}{2a} & -\frac{1}{2d} & \frac{1}{2} \\ \frac{ad-ac+bd}{ad}\frac{1}{2e} & \frac{1}{2a} & -\frac{1}{2d} & \frac{1}{2} \\ -\frac{c}{d}\frac{1}{e} & 0 & \frac{1}{d} & 0 \end{bmatrix} = A$

$\frac{ad-ac+bd}{ad} = 1 + \frac{bd-ac}{ad} = 1 + \frac{\sigma_B^2\sigma_{UJ}-\sigma_B^3\sigma_{DJ}}{\sigma_{UJ}\sigma_{DJ}}\sqrt{h} = 1 + \left(\frac{\sigma_B^2}{\sigma_{DJ}} - \frac{\sigma_B^3}{\sigma_{UJ}}\right)\sqrt{h}$

$$\begin{bmatrix} \widetilde{p}_{DJ} \\ \widetilde{p}_D \\ \widetilde{p}_U \\ \widetilde{p}_{UJ} \end{bmatrix} = A \begin{bmatrix} (r-\mu^1)\,h \\ (r-\mu^2)\,h \\ (r-\mu^3)\,h \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sigma_B^2}{\sigma_{DJ}}\left[\frac{\mu^2-r}{\sigma_B^2} - \frac{\mu^1-r}{\sigma_B^1}\right]h \\ \frac{1}{2} + \frac{h}{2}\left[\frac{\mu^3-r}{\sigma_{UJ}} - \frac{\mu^2-r}{\sigma_{DJ}} + \left(1+\left(\frac{\sigma_B^2}{\sigma_{DJ}} - \frac{\sigma_B^3}{\sigma_{UJ}}\right)\sqrt{h}\right)\frac{\mu^1-r}{\sigma_B^1}\right] \\ \frac{1}{2} + \frac{h}{2}\left[\frac{\mu^3-r}{\sigma_{UJ}} - \frac{\mu^2-r}{\sigma_{DJ}} - \left(1+\left(\frac{\sigma_B^2}{\sigma_{DJ}} - \frac{\sigma_B^3}{\sigma_{UJ}}\right)\sqrt{h}\right)\frac{\mu^1-r}{\sigma_B^1}\right] \\ \frac{\sigma_B^3}{\sigma_{UJ}}\left[\frac{\mu^1-r}{\sigma_B^1} - \frac{\mu^3-r}{\sigma_B^3}\right]h \end{bmatrix}$$

This is the equivalent martingale measure, and, hence the stochastic discount factor would be:

$$X_{t+h} = \begin{cases} \frac{\sigma_B^2}{\sigma_{DJ}}\left[\frac{\mu^2-r}{\sigma_B^2} - \frac{\mu^1-r}{\sigma_B^1}\right]\frac{2}{\lambda} & \text{if the state is DJ} \\ \frac{1+\left[\frac{\mu^3-r}{\sigma_{UJ}} - \frac{\mu^2-r}{\sigma_{DJ}} + \left(1+\left(\frac{\sigma_B^2}{\sigma_{DJ}} - \frac{\sigma_B^3}{\sigma_{UJ}}\right)\sqrt{h}\right)\frac{\mu^1-r}{\sigma_B^1}\right]h}{1-\lambda h} & \text{if the state is D} \\ \frac{1+\left[\frac{\mu^3-r}{\sigma_{UJ}} - \frac{\mu^2-r}{\sigma_{DJ}} - \left(1+\left(\frac{\sigma_B^2}{\sigma_{DJ}} - \frac{\sigma_B^3}{\sigma_{UJ}}\right)\sqrt{h}\right)\frac{\mu^1-r}{\sigma_B^1}\right]h}{1-\lambda h} & \text{if the state is U} \\ \frac{\sigma_B^3}{\sigma_{UJ}}\left[\frac{\mu^1-r}{\sigma_B^1} - \frac{\mu^3-r}{\sigma_B^3}\right]\frac{2}{\lambda} & \text{if the state is UJ} \end{cases}$$

All the probabilities are positive and the SDF is well-defined, if $\frac{\mu^2-r}{\sigma_B^2} > \frac{\mu^1-r}{\sigma_B^1} > \frac{\mu^3-r}{\sigma_B^3}$, h - small enough. First, assume, $\sigma_{UJ} = \sigma_{DJ} = J \quad \sigma_B^1 = 1, \quad \sigma_B^2 = \sigma_B^3 = \sigma, \quad \mu^1 = \mu, \quad \mu^2 = \mu + p, \quad \mu^3 = \mu - c$

$$\begin{bmatrix} \widetilde{p}_{DJ} \\ \widetilde{p}_D \\ \widetilde{p}_U \\ \widetilde{p}_{UJ} \end{bmatrix} = \begin{bmatrix} \frac{\sigma}{J}ph \\ \frac{1}{2} + \frac{h}{2}\left[\frac{-c-p}{J} + \left(1+\left(\frac{\sigma}{J}-\frac{\sigma}{J}\right)\sqrt{h}\right)(\mu-r)\right] \\ \frac{1}{2} + \frac{h}{2}\left[\frac{-c-p}{J} - \left(1+\left(\frac{\sigma}{J}-\frac{\sigma}{J}\right)\sqrt{h}\right)(\mu-r)\right] \\ \frac{\sigma}{J}ch \end{bmatrix} = \begin{bmatrix} \frac{\sigma}{J}ph \\ \frac{1}{2} + \frac{h}{2}\left[\frac{-c-p}{J} + (\mu-r)\right] \\ \frac{1}{2} + \frac{h}{2}\left[\frac{-c-p}{J} - (\mu-r)\right] \\ \frac{\sigma}{J}ch \end{bmatrix}$$

$$X_{t+h} = \begin{cases} \frac{2p}{\lambda}\frac{\sigma}{J} & \text{if the state is DJ} \\ \frac{1-\frac{(c+p)}{J}h+(\mu-r)h}{1-\lambda h} & \text{if the state is D} \\ \frac{1-\frac{(c+p)}{J}h-(\mu-r)h}{1-\lambda h} & \text{if the state is U} \\ \frac{2c}{\lambda}\frac{\sigma}{J} & \text{if the state is UJ} \end{cases}$$

We can interpret the four assets as a risk-free asset (N4), risky stock (N1), a put option on the stock (N2) and a call option on the stock (N3). Call is exercised if there is a jump up in the stock price and the investors would like to buy the stock at the original price, Put is executed if there is a jump down in the stock price and the investors would like to sell the stock at the original price. Assume for simplicity that the size of both jumps $J = 1$. Then, p - is the price of a put, c - is the price of a call. Both are payed when there is no jump, hence the SDF is proportional to $\pm(\mu - r) - c - p$. If there is a jump down, the SDF is proportional to the price of the put, and if there is a jump up, the SDF is proportional to the price of a call.