Online Appendix to: "The Labor Wedge as a Matching Friction."

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1. Appendix C

In the following subsection we show the solution to the same model as in the main body of the paper but when solved as a social planner's problem. The decentralized version of the model has a missing equilibrium condition that is typically replaced with a Nash bargaining condition to fix the real wage. We take advantage of this missing condition, and by comparing the social planner's solution with the decentralized version of the model, we construct a time-varying bargaining shock, which implicitly determines the wage rate.

1.1. The Social Planner's Problem

To compare competitive allocations with an efficient one, we solve the social planning problem. The social planner maximizes the discounted present value of the utility function:

$$\max_{\{C_t, L_t, V_t, U_{t,}, K_{t+1}\}} E_t \sum_{t=0}^{\infty} \beta^t U\left(C_t, L_t, V_t, U_{t,}\right) \tag{1}$$

subject to

$$C_t + \frac{1}{T_t}\left(K_{t+1} - (1-\delta_K)K_t\right) + G_t \leq A_t F(K_t, L_t) \tag{2}$$

$$L_t = (1-\delta_L)L_{t-1} + M_t \tag{3}$$

The optimality conditions of the planner are given by:

$$\frac{1}{T_t} = \beta E_t \frac{U'_{C_{t+1}}}{U'_{C_t}} \left(A_{t+1} F'_{K_{t+1}}(K_t, L_t) + \frac{1}{T_t}\left(1-\delta_K\right) \right) \tag{4}$$

$$A_t F'_{L_t}(K_t, L_t) + \frac{U'_{L_t}}{U'_{C_t}} = \Gamma_t - \beta E_t \left(\frac{U'_{C_{t+1}}}{U'_{C_t}} \Gamma_{t+1} \left(1-\delta_{Lt}\right) \right) \tag{5}$$

$$-\frac{U'_{V_t}}{U'_{C_t}} = \Gamma_t \frac{\partial M_t}{\partial V_t} \tag{6}$$

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**The views expressed in this paper are those of the authors and do not necessarily reflect the views of the Federal Reserve Bank of Dallas or the Federal Reserve System.

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$$-\frac{U'_{U_t}}{U'_{C_t}} = \Gamma_t \frac{\partial M_t}{\partial U_t} \tag{7}$$

Together with equations (7)-(8) they describe the allocations a social planner would choose. Γ_t is the Lagrange multiplier associated with the labor accumulation constraint. Given that T_t, A_t, δ_{Lt}, G_t and B_t are exogenous, we have a system of eight equations and eight unknowns $\{K_{t+1}, L_t, C_t, M_t, Y_t, V_t, U_t, \Gamma_t\}$.

1.2. Constructing a Time Varying Bargaining Shock

By comparing the social planner's optimality conditions with those of the decentralized problem, we can find the necessary assumptions to make the decentralized problem efficient.

By putting equations (12) and (13) in the main text together, we get that $-\frac{U'_{V_t}}{U'_{C_t}} = \eta_t \frac{M_t}{V_t}$, and if we compare this expression with equation (6), and equation (14) in the main text with equation (7), we need

$$\eta_t \frac{M_t}{V_t} = \Gamma_t \frac{\partial M_t}{\partial V_t} \tag{8}$$

$$\mu_t \frac{M_t}{U_t} = \Gamma_t \frac{\partial M_t}{\partial U_t} \tag{9}$$

so that the optimality conditions on vacancies and unemployment are the same in the decentralized and planner's problem. Furthermore, if we assume that the matching function has constant returns to scale

$$\frac{\partial M(U_t, V_t)}{\partial U_t} U_t + \frac{\partial M(U_t, V_t)}{\partial V_t} V_t = M(U_t, V_t) \tag{10}$$

then $\Gamma_t = \mu_t + \eta_t$ and the decentralized outcome is Pareto optimal. Hence, the Hosios condition for efficiency is given by:

$$\eta_t = \Gamma_t \frac{\partial M_t}{\partial V_t} \frac{V_t}{M_t} \tag{11}$$

$$\mu_t = \Gamma_t \frac{\partial M_t}{\partial U_t} \frac{U_t}{M_t} \tag{12}$$

As an illustration, assume $\frac{\partial M_t}{\partial U_t} \frac{U_t}{M_t} = \theta$ and $\frac{\partial M_t}{\partial V_t} \frac{Vt}{M_t} = (1 - \theta)$, which together with conditions (11) and (12) give

$$\eta_t = (1 - \theta)\Gamma_t \tag{13}$$

and

$$\mu_t = \theta \Gamma_t \tag{14}$$

Notice that if we replace equations (13) and (14) in equations (10) and (11) in the main text and sum them up, we get equation (5), hence the optimality conditions for labor in the decentralized version become equal to the optimality condition for labor in the planner's problem.

Furthermore, if we divide equation (10) by equation (11) in the main text and use equations (13) and (14), we get

$$\frac{w_t + \frac{U'_{L_t}}{U'_{C_t}}}{A_t F'_{L_t}(K_t, L_t) - w_t} = \frac{\mu_t - \beta E_t \left(\frac{U'_{C_{t+1}}}{U'_{C_t}} \mu_{t+1} \left(1 - \delta_{Lt+1}\right) \right)}{\eta_t - \beta E_t \left(\frac{U'_{C_{t+1}}}{U'_{C_t}} \eta_{t+1} \left(1 - \delta_{Lt+1}\right) \right)} = \frac{\theta}{1-\theta} \tag{15}$$

Given that w_t is the wage earned by the worker and $-\frac{U'_{L_t}}{U'_{C_t}}$ is his reservation utility, the term $w_t + \frac{U'_{L_t}}{U'_{C_t}}$ represents the instantaneous benefit from the match earned by the worker. Since the bargaining power of the worker is constant and equal to θ, the optimal wage rate satisfies

$$\left(w_t + \frac{U'_{L_t}}{U'_{C_t}} \right) = \theta \left(A_t F'_{L_t}(K_t, L_t) + \frac{U'_{L_t}}{U'_{C_t}} \right) \tag{16}$$

where $A_t F'_{L_t}(K_t, L_t) + \frac{U'_{L_t}}{U'_{C_t}}$ is the difference between the marginal product of labor and the marginal disutility of labor. This term represents the instantaneous marginal value of the match, and a fraction θ goes to the worker.

To introduce the time-varying bargaining shock, we build on this result, re-parameterize and substitute θ by ϕ_t. ϕ_t is time-varying and follows an exogenous autoregressive process. Notice that replacing θ by ϕ_t implies that allocations are suboptimal whenever $\phi_t \neq \theta$.

Equations (13) and (14) are replaced by

$$\eta_t = (1 - \phi_t)\Gamma_t \tag{17}$$

$$\mu_t = \phi_t \Gamma_t \tag{18}$$

Once again, if we substitute equations (17) and (18) into (10) and (11) of the main text, we get equation (5) so it is still true that the optimality conditions for labor of the decentralized version imply the optimality condition for labor of the planner's problem. Dividing equation (10) by equation (11) and using equations (17) and (18), we get

$$\frac{w_t + \frac{U'_{L_t}}{U'_{C_t}}}{A_t F'_{L_t}(K_t, L_t) - w_t} = \frac{\phi_t \Gamma_t - \beta E_t \left(\frac{U'_{C_{t+1}}}{U'_{C_t}} \phi_{t+1} \Gamma_{t+1} \left(1 - \delta_{Lt+1}\right) \right)}{(1-\phi_t)\Gamma_t - \beta E_t \left(\frac{U'_{C_{t+1}}}{U'_{C_t}} (1-\phi_{t+1}) \Gamma_{t+1} \left(1 - \delta_{Lt+1}\right) \right)} \tag{19}$$

Hence, the optimal wage rate satisfies

$$\left(w_t + \frac{U'_{L_t}}{U'_{C_t}} \right) = \frac{\phi_t - \beta E_t \left(\frac{U'_{C_{t+1}}}{U'_{C_t}} \phi_{t+1} \Psi_{t+1} \right)}{1 - \beta E_t \left(\frac{U'_{C_{t+1}}}{U'_{C_t}} \Psi_{t+1} \right)} \left(A_t F'_{L_t}(K_t, L_t) + \frac{U'_{L_t}}{U'_{C_t}} \right) \tag{20}$$

where $\Psi_{t+1} = \frac{\Gamma_{t+1}}{\Gamma_t} \left(1 - \delta_{Lt+1}\right)$ and can be interpreted as a stochastic discount factor for labor.

This wage-setting rule is general enough to capture various mechanisms of wage adjustment, including wage rigidity proposed by Hall (2005). For example, if we denote $f(\phi_t)$ the first term on the right hand side of equation (20), then the recursive formulation for the bargaining shock

$$f(\phi_t) = (1-\varrho)\frac{w_{t-1} + \frac{U'_{L_t}}{U'_{C_t}}}{A_t F'_{L_t}(K_t, L_t) + \frac{U'_{L_t}}{U'_{C_t}}} + \varrho\theta$$

is equivalent to the partial adjustment wage setting rule proposed by Hall:

$$w_t = (1-\varrho) w_{t-1} + \varrho w_t^{Nash}.$$

1.3. Identification

In this section we show how, given data on allocations (output, investment, consumption, employment, vacancies and unemployment), one can solve for the shocks. Let us first rewrite the equations of the model given the parametric assumptions and functional forms used in the paper:

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha} \tag{21}$$

$$X_t = K_{t+1} - (1-\delta_K) K_t \tag{22}$$

$$C_t + \frac{X_t}{T_t} + G_t = Y_t \tag{23}$$

$$L_t = (1-\delta_{Lt}) L_{t-1} + B_t U_t^{\theta} V_t^{1-\theta} \tag{24}$$

$$\frac{1}{T_t} = \beta E_t \frac{C_t}{C_{t+1}}\left(\alpha\frac{Y_{t+1}}{K_{t+1}} + \frac{(1-\delta_K)}{T_{t+1}}\right) \tag{25}$$

$$\mu_t = w_t - C_t\chi (L_t + U_t + V_t)^{\gamma} + \beta E_t \frac{C_t}{C_{t+1}}\mu_{t+1}(1-\delta_{Lt+1}) \tag{26}$$

$$\eta_t = (1-\alpha)\frac{Y_t}{L_t} - w_t + \beta E_t \frac{C_t}{C_{t+1}}\eta_{t+1}(1-\delta_{Lt+1}) \tag{27}$$

$$C_t\chi (L_t + U_t + V_t)^{\gamma} = \eta_t B_t \left(\frac{U_t}{V_t}\right)^{\theta} \tag{28}$$

$$C_t\chi (L_t + U_t + V_t)^{\gamma} = \mu_t B_t \left(\frac{V_t}{U_t}\right)^{1-\theta} \tag{29}$$

$$\frac{\eta_t}{\mu_t} = \frac{1-\phi_t}{\phi_t} \tag{30}$$

Now we shall describe a mechanism to recover the shocks given parameters and functional forms. Given data on consumption C_t (or government spending G_t), output Y_t, investment X_t, employment $L_t + V_t$, number of vacancies V_t and the unemployment rate $\frac{U_t}{L_t+V_t+U_t}$, one can uniquely recover the time path for the variables of interest L_t,V_t,U_t. Then equation (22) uniquely pins down the path for capital given the initial level K_0, equation (21) pins down the efficiency shock A_t, equation (23) pins down consumption or government spending, and equation (25) can be solved forward to obtain the path for the investment shock as in CKM.

From equations (28) and (29) it follows that $\eta_t U_t = \mu_t V_t$. Then, summing up equations (26) and (27), one obtains:

$$\begin{aligned} & -C_t \chi \left(L_t + U_t + V_t\right)^{\gamma} + (1-\alpha)\frac{Y_t}{L_t} \\ = \mu_t\left(1+\frac{V_t}{U_t}\right) & - \beta E_t \frac{C_t}{C_{t+1}} \mu_{t+1}\left(1+\frac{V_{t+1}}{U_{t+1}}\right)\left(1-\delta_{Lt+1}\right) \end{aligned} \tag{31}$$

Using equation (28) the Lagrange multiplier μ_t can be expressed as a function of the matching shock B_t:

$$\mu_t = \frac{\chi \left(L_t + U_t + V_t\right)^{\gamma}}{B_t \left(\frac{V_t}{U_t}\right)^{1-\theta}} \tag{32}$$

Also the separation rate is connected to the matching shock through the labor accumulation equation (24):

$$\left(1-\delta_{Lt+1}\right) = \frac{L_{t+1} - B_{t+1} U_{t+1}^{\theta} V_{t+1}^{1-\theta}}{L_t} \tag{33}$$

Then, substituting equations (32) and (33) into equation (31), we obtain:

$$\begin{aligned} \left(\frac{1+\frac{V_t}{U_t}}{\left(\frac{V_t}{U_t}\right)^{1-\theta}}\frac{1}{B_t} - 1\right) L_t &= \frac{(1-\alpha) Y_t}{C_t \chi \left(L_t + U_t + V_t\right)^{\gamma}} \\ + \beta E_t \frac{C_t}{C_{t+1}} \frac{1+\frac{V_{t+1}}{U_{t+1}}}{\left(\frac{V_{t+1}}{U_{t+1}}\right)^{1-\theta}} & \left(\frac{L_{t+1} + U_{t+1} + V_{t+1}}{L_t + U_t + V_t}\right)^{\gamma} \left[\frac{L_{t+1}}{B_{t+1}} - U_{t+1}^{\theta} V_{t+1}^{1-\theta}\right] \end{aligned} \tag{34}$$

Equation (34) provides a forward-looking equation for the matching shock B_{t+1} as a function of B_t. Solving this equation recursively given some initial value B_0 and making assumptions about expectation formation, we can recover the whole path for the matching shock.[1] Then equation (33) allows us to back up the separation rate and equations (29) and (28) allow us to calculate the Lagrange multipliers μ_t and η_t. Then, from equation (30), we can compute the bargaining shock ϕ_t.

All together, equations (22-30) describe a one-to-one mapping between the data and the underlying shocks. However the algorithm described here is hard to implement directly for two reasons. First, the equations are forward-looking and can only be solved under certain assumptions about expectation formation. Second, many of the parameters of the model are unknown and cannot be simply calibrated from microeconomic data. That is the reason why we postulate stochastic processes for the shocks, linearize the model around a steady-state to compute an approximate solution, and use the Kalman filter to recover the underlying processes for the shocks.

2. Appendix D

2.1. The Detrended Model

Once we detrend all the variables of the model, we come to the following representation:

$$E_t \psi_{t+1} \left(\alpha \frac{y_{t+1}}{k_{t+1}} - \frac{1-\delta_K}{\tau_{t+1}} \right) = 1$$

$$y_t = a_t k_t^{\alpha} L_t^{1-\alpha}$$

$$c_t + z_t k_{t+1} - (1-\delta_K) \frac{k_t}{\tau_t} + g_t = y_t$$

$$\Gamma_t = \left((1-\alpha) \frac{y_t}{L_t} - \kappa_t \right) + E_t \psi_{t+1} \Gamma_{t+1} (1-\delta_{Lt})$$

$$\left(B_t U_t^{\theta} V_t^{1-\theta} \right) \Gamma_t = (V_t + U_t) \kappa_t$$

$$\phi_t V_t = (1-\phi_t) U_t$$

$$L_t = (1-\delta_{Lt}) L_{t-1} + B_t U_t^{\theta} V_t^{1-\theta}$$

$$z_t^{1-\alpha} = a_t \tau_t^{\alpha}$$

[1] In the first step, we assume a diagonal VAR structure for the shocks which allows us to pin down the expectations. We estimate the VAR structure in the second step.

$$m_t = (1 - \delta_{Lt}) \frac{\Gamma_t}{\Gamma_{t-1}}$$

$$\psi_t = \beta \left(\frac{c_{t-1}}{c_t} \right) \frac{1}{z_{t-1}}$$

$$\kappa_t = \chi c_t \left(L_t + U_t + V_t \right)^{\gamma}$$

$$q_t = (1 - \phi_t) \Gamma_t B_t \left(\frac{U_t}{V_t} \right)^{\theta}$$

$$(1 - E_t m_{t+1} \psi_{t+1}) (w_t - \kappa_t) = (\phi_t - E_t m_{t+1} \psi_{t+1} \phi_{t+1}) \left((1 - \alpha) \frac{y_t}{L_t} - \kappa_t \right)$$

$$X_t = \begin{bmatrix} \log \delta_{Lt} - \log \delta_{Lss} \\ \log \phi_t - \log \phi_{ss} \\ \log B_t - \log B_{ss} \\ \log a_t - \log a_{ss} \\ \log \tau_t - \log \tau_{ss} \\ \log g_t - \log g_{ss} \end{bmatrix}$$

$$X_t = \begin{bmatrix} \rho_S & & & & & \\ & \rho_B & & & & \\ & & \rho_M & & & \\ & & & \rho_A & & \\ & & & & \rho_T & \\ & & & & & \rho_G \end{bmatrix} X_{t-1} + Q \begin{bmatrix} \sigma_S & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_B & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_M & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_A & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_T & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_G \end{bmatrix} \varepsilon_t$$

$$d \log GDP_t = \log \frac{y_t + q_t V_t}{y_{t-1} + q_{t-1} V_{t-1}} z_{t-1}$$

$$d \log Cons_t = \log \frac{c_t}{c_{t-1}} z_{t-1}$$

$$d \log Inv_t = \log \frac{k_{t+1} z_t \tau_t - (1 - \delta_K) k_t}{k_t z_{t-1} \tau_{t-1} - (1 - \delta_K) k_{t-1}} z_{t-1} \tau_{t-1}$$

$$Hours_t = \frac{L_t + V_t}{L_{ss} + V_{ss}}$$

$$Unemp_t = \frac{U_t}{L_t + V_t + U_t}.$$

$$HWant_t = \frac{V_t}{V_{ss}}$$

2.2. Computing the Steady-State.

Choose a value of L_{ss}

1) $z_{ss} = (a_{ss}\tau_{ss}^{\alpha})^{\frac{1}{1-\alpha}}$

2) Denote $\varphi = \left(\left(\frac{z_{ss}}{\beta} + \frac{1-\delta_K}{\tau_{ss}}\right)/\alpha a_{ss}\right)^{-\frac{1}{1-\alpha}}$

3) $k_{ss} = \varphi L_{ss}$ $\quad y_{ss} = a_{ss}\varphi^{\alpha} L_{ss}$

4) $c_{ss} = \left[(1-g_{ss})\, a_{ss}\varphi^{\alpha} - \left(z_{ss} - \frac{(1-\delta_K)}{\tau_{ss}}\right)\varphi\right] L_{ss}$

5) $B_{ss} = \frac{1}{\omega_{ss}}\left(\frac{\phi_{ss}}{1-\phi_{ss}}\right)^{1-\theta}$

6) $U_{ss} = \omega_{ss}\delta_L L_{ss}$ $\quad V_{ss} = \frac{1-\phi_{ss}}{\phi_{ss}} U_{ss}$

7) $\xi = \frac{y_{ss}}{L_{ss}c_{ss}} \frac{1-\alpha}{\left(1+\frac{\omega_{ss}}{\phi_{ss}}\left(1-\frac{\beta}{z_{ss}}(1-\delta_L)\right)\right)}$

8) We have assumed a normalization $\chi = \frac{\xi}{(L_{ss}+U_{ss}+V_{ss})^{\gamma}}$

9) $\kappa_{ss} = \xi c_{ss}$ $\quad m_{ss} = 1-\delta_L$

10) $\Gamma_{ss} = \xi c_{ss}\frac{\omega_{ss}}{\phi_{ss}}$

11) $w_{ss} = \phi_{ss}(1-\alpha)\frac{y_{ss}}{L_{ss}} - (1-\phi_{ss})\,\kappa_{ss}$

12) $\psi_{ss} = \frac{\beta}{z_{ss}}$ $\quad q_{ss} = \kappa_{ss}$