Macro Comp Fall 2003 Questions 1-3 Anton Cheremukhin

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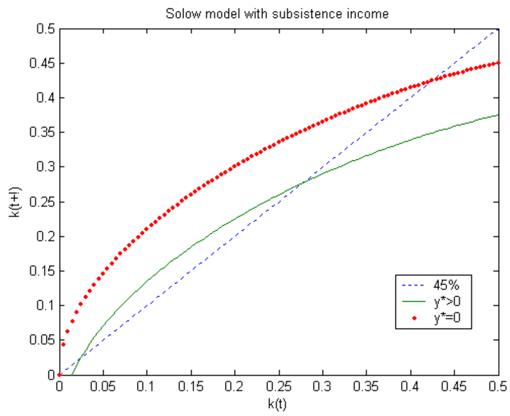
Exercise 1 *Solow model*

The model is described by four equations:
$y_t = f(k_t)$, $y_t = c_t + i_t$, $k_{t+1} = (1-\delta)k_t + i_t$, $i_t = s(y_t - y^*)$
This implies the law of motion of capital: $k_{t+1} = (1-\delta)k_t + s(f(k_t) - y^*)$



There are three equilibria: $\{0, \underline{k}, \overline{k}\}$. The 0 and $\overline{k}$ are stable. There is a region around zero, which is very similar to a "poverty trap".

Exercise 2 *Kinked production*

Consumer: $U = \Sigma\beta^t c_t \to \max$ s.t. $c_t + k_{t+1} = r_t k_t$, k_0
Firm: $\pi = y_t - r_t\widetilde{k}_t \to \max$ s.t. $y_t = f\left(\widetilde{k}_t\right)$
S.E.: Allocations $\left\{c_t, k_{t+1}, \widetilde{k}_t\right\}_{t=0}^{\infty}$ and Prices$\{r_t\}_{t=0}^{\infty}$ s.t.
a) A are optimal given P b) markets clear: $k_t = \widetilde{k}_t$, $c_t + k_{t+1} = y_t$

$$f(k) = \begin{cases} Ak + (B-A)\overline{k}, & k > \overline{k} \\ Bk, & k \leq \overline{k} \end{cases} \qquad A < \frac{1}{\beta} < B$$

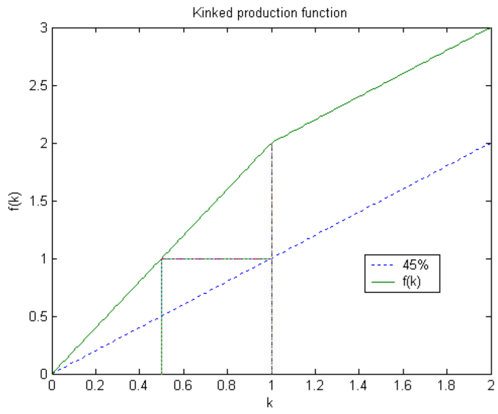
Bellman: $v(k) = \max_{k'}[f(k) - k' + \beta v(k')] = \max_{k'} \begin{bmatrix} Ak + c - k' + \beta v(k'), & k > \overline{k} \\ Bk - k' + \beta v(k'), & k \leq \overline{k} \end{bmatrix}$

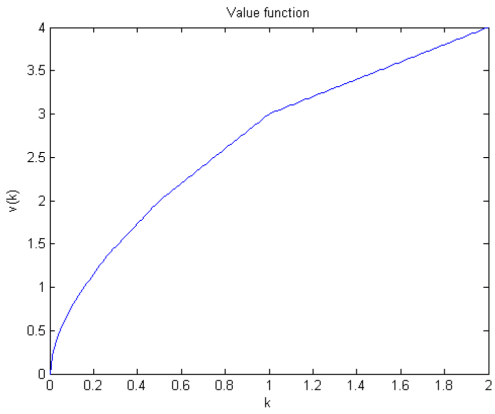
FOC: $v'(k') = \frac{1}{\beta}$ $\quad v'(k) = \begin{bmatrix} A, & k > \overline{k} \\ B, & k < \overline{k} \\ [A, B], & k = \overline{k} \end{bmatrix}$ $\quad s.s.: k = \overline{k}$.

The guys want to choose tomorrows capital to be $\overline{k}$ if they can.

Policy function: $k' = \begin{bmatrix} \overline{k}, & k > \overline{k}/B \\ Bk, & k \leq \overline{k}/B \end{bmatrix}$. Define $U_0 = v\left(\overline{k}\right) = \frac{(B-1)\overline{k}}{1-\beta}$.

Value function: $v'(k) = \begin{bmatrix} \beta^T\left(kB^{T+1} - \overline{k} + \beta U_0\right), & \overline{k}/B < kB^T < \overline{k} \\ Bk - \overline{k} + \beta U_0, & \overline{k}/B \leq k < \overline{k} \\ A\left(k - \overline{k}\right) + U_0, & \overline{k} \leq k \end{bmatrix}$





Exercise 3 *Business cycle volatility and runup in stock prices*

Consider a representative agent economy. The RA has separable CRRA utility. There is a single Lucas tree that pays off the aggregate endowment every period. The growth rate of the aggregate endowment $\Delta \ln C_t$ is i.i.d. and normally distributed with mean μ and standard deviation σ.

(.) State the sequential and recursive problems

(i) Compute the price-dividend ratio in closed form.

(ii) Compute the riskless rate in closed form.

(iii) Some people have argued that the recent runup in stock prices is due to a drop in business cycle volatility. In the model, what happens to prices and interest rates when σ falls? Interpret the result. Does the argument make sense?

Solution.

SeqCE: Allocations $\{c_t, s_{t+1}, b_{t+1}\}$ and Prices $\{1, q_t, p_t\}$ s.t.

(1) A solve: $\max_{c_t, s_{t+1}, b_{t+1}} \{E\sum_{t=0}^{\infty} \beta^t u(c_t) \,|b_t + s_t(q_t + d_t) \geq c_t + q_t s_{t+1} + p_t b_{t+1}\}$ given P.

(2) P satisfy Market Clearing (corresponding price):

Consumption $c_t = d_t | (1)$ Risky equity: $s_t = 1 | (q_t)$ Riskless bond: $b_t = 0 | (p_t)$

(3) Dividends d_t are exogenous and their distribution is known.

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Bellman: $V(s, b, S, B, x) = \max_{s', b'} \left[u(b + s(q + d(x)) - qs' - pb') + \beta E_{x'|x} V(s', b', S', B', x')\right]$

RCE: (1) Value function: $V(s, b, S, B, x)$

(2) Decision Rules $s'(s, b, S, B, x), b'(s, b, S, B, x)$

(3) Price Functions $q(s, b, S, B, x), p(s, b, S, B, x)$

(4) Laws of Motion of aggregate holdings $H_s(S, B, x), H_k(S, B, x)$.

s.t.

(i)DRs solve Bellman given PFs and LMs.

(ii)PFs satisfy market clearing given the state (S, B, x): $s' = S' = 1,\quad b = B' = 0$.

(iii)Perceptions are correct: $H_s(S, B, x) = s'(S, B, S, B, x)$, $H_b(S, B, x) = b'(S, B, S, B, x)$

FOC: $qu'_c = \beta E_{x'|x} V'_s(s', b', S', B', x') \qquad pu'_c = \beta E_{x'|x} V'_b(s', b', S', B', x')$

Envelope: $V'_s(s, b, S, B, x) = (q + d(x)) u'_c \qquad V'_b(s, b, S, B, x) = u'_c$

Euler: $qu'_c = \beta E_{x'|x}[(q' + d(x')) u'_{c'}] \qquad pu'_c = \beta E_{x'|x}[u'_{c'}]$

Asset prices: $q = \beta E_{x'|x}\left[\frac{u'_{c'}}{u'_c}(q' + d(x'))\right] \qquad p = \beta E_{x'|x}\left[\frac{u'_{c'}}{u'_c}\right]$

Now use the fact that $u(c) = \frac{d^{1-\rho}}{1-\rho}$. Then $\frac{u'_{c'}}{u'_c} = \left(\frac{c}{c'}\right)^{\rho}$. Also, remember, that $c = d$.

$p_t = \beta E_t\left[\left(\frac{c_t}{c_{t+1}}\right)^{\rho}\right] \qquad q_t = \beta E_t\left[\left(\frac{c_t}{c_{t+1}}\right)^{\rho}(q_{t+1} + c_{t+1})\right]$

Can solve recursively for the equity price:

$$q_t = \beta E_t\left[\left(\frac{c_t}{c_{t+1}}\right)^{\rho}(q_{t+1}+c_{t+1})\right] = \beta E_t\left[\left(\frac{c_t}{c_{t+1}}\right)^{\rho}\left(\beta\left(\frac{c_{t+1}}{c_{t+2}}\right)^{\rho}(q_{t+2}+c_{t+2})+c_{t+1}\right)\right] =$$

$$= E_t\left[\beta\left(\frac{c_t}{c_{t+1}}\right)^{\rho}c_{t+1}+\beta^2\left(\frac{c_t}{c_{t+2}}\right)^{\rho}(q_{t+2}+c_{t+2})\right] = ...$$

Use the transversality conditions: $\lim_{j\to\infty}\beta^j\left(\frac{c_t}{c_{t+j}}\right)^{\rho}q_{t+j}=0.$

Then the **closed forms** for the prices are: $p_t = E_t\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\rho}$.

$$q_t = E_t\sum_{j=1}^{\infty}\beta^j c_t^{\rho}c_{t+j}^{1-\rho} = E_t\sum_{j=1}^{\infty}\beta^j c_t\left(\frac{c_{t+j}}{c_t}\right)^{1-\rho} = E_t\sum_{j=1}^{\infty}\beta^j c_t\left(\left(\frac{c_{t+1}}{c_t}\right)\left(\frac{c_{t+2}}{c_{t+1}}\right)...\left(\frac{c_{t+j}}{c_{t+j-1}}\right)\right)^{1-\rho}$$

Now use the fact that $\Delta\ln c_{t+1} = \ln\frac{c_{t+1}}{c_t} = \varepsilon_{t+1}\sim N(\mu,\sigma^2)$

Hence, $\ln\left(\frac{c_{t+1}}{c_t}\right)^{-\rho} = -\rho\varepsilon_{t+1}\sim N(-\rho\mu,\rho^2\sigma^2)$ i.i.d.

$\ln\left(\frac{c_{t+1}}{c_t}\right)^{1-\rho} = (1-\rho)\,\varepsilon_{t+1}\sim N\left((1-\rho)\,\mu,(1-\rho)^2\sigma^2\right)$ i.i.d.

$\ln\left(\left(\frac{c_{t+1}}{c_t}\right)\left(\frac{c_{t+2}}{c_{t+1}}\right)...\left(\frac{c_{t+j}}{c_{t+j-1}}\right)\right)^{1-\rho} = (1-\rho)\left(\varepsilon_{t+1}+\varepsilon_{t+2}+...+\varepsilon_{t+j}\right)\sim N\left((1-\rho)\,j\mu,(1-\rho)^2 j\sigma^2\right)$

Using the fact that, $E\exp u_t = \exp\left(Eu_t+\frac{Vu_t}{2}\right)$, denote $\exp\left((1-\rho)\,\mu+\frac{(1-\rho)^2\sigma^2}{2}\right)=\varphi$

$$q_t = \sum_{j=1}^{\infty}\beta^j c_t\exp\left((1-\rho)\,j\mu+\frac{(1-\rho)^2 j\sigma^2}{2}\right) = c_t\sum_{j=1}^{\infty}(\beta\varphi)^j = c_t\frac{\beta\varphi}{1-\beta\varphi}$$

Dividend-price ratio: $\boxed{c_t/q_t = \frac{1}{\beta\exp\left((1-\rho)\mu+\frac{(1-\rho)^2\sigma^2}{2}\right)}-1}$

$$p_t = \beta E_t\left(\frac{c_{t+1}}{c_t}\right)^{-\rho} = \beta\exp\left(-\rho\mu+\frac{\rho^2\sigma^2}{2}\right) = \frac{1}{1+r}$$

Risk-free rate: $\boxed{r_t = \frac{1}{\beta}\exp\left(\rho\mu-\frac{\rho^2\sigma^2}{2}\right)-1}$

$\frac{\partial}{\partial\sigma}(r_t) = -\rho^2\frac{\sigma}{\beta}\exp\left(\rho\mu-\frac{\rho^2\sigma^2}{2}\right)<0 \qquad \frac{\partial}{\partial\sigma}(c_t/q_t) = -\frac{1}{\beta}\frac{(1-\rho)^2}{\exp\left((1-\rho)\mu+\frac{(1-\rho)^2\sigma^2}{2}\right)}\sigma<0$

These results imply that when volatility falls, the risk free rate grows and the stock prices fall.