1 Mathematics

- correspondence φ is upper semicts.: $x^q \to x^0$, $y^q \in \varphi(x^q)$, $y^q \to y^0 \Rightarrow y^0 \in \varphi(x^0)$
- correspondence φ is lower semicts.: $x^q \to x^0$, $y^0 \in \varphi(x^0) \Rightarrow \exists y^q \in \varphi(x^q)$ s.t. $y^q \to y^0$
- $f : S \times T \to \mathbb{R}$ cts., T compact, correspondence $\varphi(s) \subset T$ cts. $\Rightarrow \eta(s) := \{t^\star \in \varphi(s) : f(s,t^\star) = \max_{t\in\varphi(s)} f(s,t)\}$ upper semicts. at s, $g(s) = f(s,\eta(s))$ cts. at s
- Kakutani: $S \subset \mathbb{R}^m$ non-empty, compact, convex, φ upper semicts. correspondence from S to S s.t. $\varphi(x)$ non-empty, convex $\forall x \in S \Rightarrow \exists$ fixed point
- positively semi-independent: cones $C_1, \ldots, C_n$, vertex 0 s.t. $x_j \in C_j$ and $\sum x_j = 0 \Rightarrow x_j = 0$
- asymptotic cone $\mathbf{A}S$ of $S \subset R^m$
  - $S^k := \{x \in S : |x| \geqslant k\}$
  - $\Gamma(S^k) :=$ least closed cone containing S^k, vertex 0
  - $\mathbf{A}S := \bigcap_{k\geqslant 0} \Gamma(S^k)$
  - S bounded $\Rightarrow \mathbf{A}S = \{0\}$
  - $\mathbf{A}(\{x\} + S) = \mathbf{A}S \subset \mathbf{A}(S + T)$, $\mathbf{A}(\prod S_j) \subset \prod \mathbf{A}S_j$
  - $\bigcap \mathbf{A}S_j = \{0\} \Rightarrow \bigcap S_j$ bounded
  - S_i closed, $\mathbf{A}S_i$ pos. semi-indep. $\Rightarrow \sum S_i$ closed
  - S closed, convex, $0 \in S \Rightarrow \mathbf{A}S \subset S$

2 Commodities and Prices

- commodities
  - good/service + date + location
  - commodity space $\mathbb{R}^l$
  - action of an economic agent = point $a \in \mathbb{R}^l$
- prices
  - price system $p \in \mathbb{R}^l$
  - value of an action $a = a \cdot p$

3 Producers

- production/supply $y_j \in \mathbb{R}^l$, outputs $\geqslant 0$, inputs $\leqslant 0$
- production set $Y_j \subset \mathbb{R}^l$
- total production $y := \sum_{j=1}^n y_j$
- total production set $Y := \sum_{j=1}^n Y_j$
- assumptions on production sets
  - Y_j closed, $\boxed{Y \text{ closed}}$
  - $\boxed{0 \in Y_j}$ (possibility of inaction), $0 \in Y$
  - $Y \cap \Omega \subset \{0\}$ (impossibility of free production)
  - $\boxed{Y \cap (-Y) \subset \{0\}}$ (irreversibility)
  - Y_j convex, $\boxed{Y \text{ convex}}$
  - non-decreasing -, non-increasing -, constant returns to scale
  - $\boxed{Y \supset (-\Omega)}$ (free disposal)
- Y closed, convex, $Y \supset (-\Omega) \Rightarrow Y \supset (Y - \Omega)$
- Y_j closed, convex, $Y \cap (-Y) = \{0\} \Rightarrow Y$ closed
- profit maximization
  - profit $p \cdot y_j$, total profit $p \cdot y$
  - prices given, choose $y_j \in Y_j$ to maximize profit $\to$ equilibrium production
  - $T_j' := \{p \in \mathbb{R}^l : p \cdot Y_j \text{ has max.}\}$ cone with vertex 0
  - $\eta_j : T_j' \twoheadrightarrow Y$, $\eta_j(p) := \{y_j \in Y_j : p \cdot y_j = \max p \cdot Y_j\}$, supply correspondence
  - $\pi_j : T_j' \to \mathbb{R}, \pi_j(p) := \max p \cdot Y_j$, profit function
  - $\eta : \bigcap T_j' \twoheadrightarrow Y$, $\eta(p) := \sum \eta_j(p)$
  - $\pi : \bigcap T_j' \to \mathbb{R}$, $\pi(p) := \sum \pi_j(p)$
  - $\eta(p) = \{y \in Y : p \cdot y = \max p \cdot Y\}$, $\pi(p) = \max p \cdot Y$
  - Y_j convex $\Rightarrow \eta_j(p)$ and $\eta(p)$ convex
  - $Y \supset (-\Omega)$: $p \in \bigcap T_j' \Rightarrow p \geqslant 0$
- price variations
  - Y_j compact $\Rightarrow \eta_j$, η upper semicts. on $T_j' = \mathbb{R}^l$, π_j, π cts. on $\mathbb{R}^l$

4 Consumers

- consumption/demand $x_i \in \mathbb{R}^l$, inputs $\geqslant 0$, outputs $\leqslant 0$
- consumption set $X_i \subset \mathbb{R}^l$
- total consumption $x := \sum_{i=1}^m x_i$
- total consumption set $X := \sum_{i=1}^m X_i$
- assumptions on consumtion sets
  - $\boxed{X_i \text{ closed,}}$ X closed
  - $\boxed{X_i}$ has a $\boxed{\text{lower bound for } \leqslant,}$ i.e., $\exists \chi_i$ s.t. $\chi_i \leqslant x_i$ $\forall x_i \in X_i$, X has a lower bound for $\leqslant$
  - X_i connected
  - $\boxed{X_i \text{ convex,}}$ X convex
- X_i closed, lower bound $\Rightarrow X$ closed
- preferences: complete preordering $\precsim_i$
- $\boxed{\nexists \text{ satiation consumption,}}$ $\nexists x_i'$ s.t. $x_i \precsim_i x_i'$ $\forall x_i \in X_i$
- continuity assumption on preferences
  - utility fct. u_i: increasing fct. from $(X_i, \precsim_i)$ to $\mathbb{R}$
  - X_i connected, $\boxed{\forall x_i' \in X_i : \{x_i \in X_i :\ x_i \precsim x_i'\}}$ and $\boxed{\{x_i \in X_i :\ x_i' \precsim x_i\} \text{ are closed}}$ $\Rightarrow \exists$ cts. utility fct.
- convexity assumptions on preferences
  - $x_i^1 \succsim_i x_i^2 \Rightarrow t x_i^1 + (1-t) x_i^2 \succsim_i x_i^2$ (weak convexity)
  - $\boxed{x_i^1 \succ_i x_i^2 \Rightarrow t x_i^1 + (1-t) x_i^2 \succ_i x_i^2}$ (convexity)
  - continuity assumption + convexity $\Rightarrow$ weak conv.
- wealth constraint
  - wealth distribution $w = (w_i) \in \mathbb{R}^m$, $p \cdot x_i \leqslant w_i$
  - $S_i := \{(p,w) \in \mathbb{R}^{l+m} :\ \exists x_i \in X_i \text{ s.t. } p \cdot x_i \leqslant w_i\}$
  - $\gamma_i : S_i \twoheadrightarrow X_i$, $\gamma_i(p,w) = \{x_i \in X_i :\ p \cdot x_i \leqslant w_i\}$
  - X_i compact, convex, $(p^0, w^0) \in S_i$, $w_i^0 \neq \min p^0 \cdot X_i \Rightarrow \gamma_i$ cts. at (p^0, w^0)
- preference satisfaction
  - given $(p,w) \in S_i$, choose greatest element wrt. $\precsim_i$ in $\gamma_i(p,w) \to$ equilibrium consumption
  - $S_i' := \{(p,w) \in S_i :\ \gamma_i(p,w) \text{ has greatest element}\}$
  - $\xi_i : S_i' \twoheadrightarrow X_i$, $\xi_i(p,w) = \{x_i \in \gamma_i(p,w) :\ x_i \text{ is greatest element of } \gamma_i(p,w)\}$

- ▷ $\xi : \bigcap S'_i \twoheadrightarrow X$, $\xi(p,w) = \sum \xi_i(p,w)$
- ▷ X_i convex, $\precsim_i$ cts., weak convexity $\Rightarrow \xi_i(p,w)$ convex
- ▷ $(p,w) \in S'_i$, x'_i greatest element of $\gamma_i(p,w)$, x'_i not satiation, $\precsim_i$ satisfies convexity $\Rightarrow p \cdot x'_i = w_i$
- ▶ price-wealth variations
  - ▷ X_i compact, cts. utility fct., γ_i cts. at $(p,w) \in S'_i = S_i \Rightarrow \xi_i$, ξ upper semicts. at (p,w)

5 Equilibrium

- ▶ economy E: $X_i \subset \mathbb{R}^l, \precsim_i, Y_j \subset \mathbb{R}^l, \omega \in \mathbb{R}^l$ (total resources)
- ▶ state of E: $((x_i),(y_j)) \in \mathbb{R}^{l(m+n)}$
- ▶ net demand $x - y$, excess demand $z = x - y - \omega$, $Z = X - Y - \{\omega\}$
- ▶ $((x_i),(y_j)) \in M$ (market equilibrium) $:\Leftrightarrow x - y = \omega$
- ▶ $((x_i),(y_j)) \in A$ (attainable state) $:\Leftrightarrow (x - y = \omega) \wedge (x_i \in X_i) \wedge (y_j \in Y_j)$
- ▶ $A = (\prod X_i) \times (\prod Y_j) \cap M$, X_i, Y_j closed/convex $\Rightarrow A$ closed/convex
- ▶ $x_i \in \widehat{X}_i$ (attainable) $:\Leftrightarrow \exists ((x_i),(y_j)) \in M$
- ▶ $y_j \in \widehat{Y}_j$ (attainable) $:\Leftrightarrow \exists ((x_i),(y_j)) \in M$
- ▶ X lower bound, Y closed, convex, $Y \cap \Omega = \{0\}$:
  (a) if $n = 1$ or $Y \cap (-Y) \subset \{0\} \Rightarrow A$ bounded
  (b) if X closed $\Rightarrow X - Y$ closed
- ▶ private ownership economy $\mathcal{E}$
  - ▷ E & $\omega_i \in \mathbb{R}^l$ & $\theta_{ij} \in \mathbb{R}_+$ s.t. $\sum \omega_i = \omega$, $\sum_i \theta_{ij} = 1$
  - ▷ equilibrium $((x^\star_i),(y^\star_j),p^\star) \in \mathbb{R}^{l(m+n+1)}$ s.t.:
    - (α) $x^\star_i$ greatest element of $\{x_i \in X_i :\ p^\star \cdot x_i \leqslant p^\star \cdot w_i + \sum_j \theta_{ij} p^\star \cdot y^\star_j\}$
    - (β) $y^\star_j$ max. profit relative to $p^\star$ on Y_j
    - (γ) $x^\star - y^\star = \omega$
- ▶ market equilibrium
  - ▷ $y_j \in \eta_j(p) \Rightarrow \pi_j(p) = p \cdot y_j \Rightarrow$ wealth distr. $p \cdot \omega_i + \sum_j \theta_{ij}\pi_j(p)$, def. $\xi'_i(p) := \xi_i(p, p\cdot\omega_i + \sum_j \theta_{ij}\pi_j(p))$
  - ▷ $C := \{p : \eta_j(p) \neq \emptyset,\ \xi'_i(p) \neq \emptyset\}$
  - ▷ $\zeta : C \twoheadrightarrow Z$, $\zeta(p) = \xi'(p) - \eta(p) - \{\omega\}$ (excess demand correspondence)
  - ▷ $\exists$ equilibrium $\Leftrightarrow \exists p \in C$ s.t. $0 \in \zeta(p)$
  - ▷ $t > 0 \Rightarrow \zeta(tp) = \zeta(p) \Rightarrow C$ cone, vertex 0, but 0 excluded $\Leftrightarrow \exists$ insatiable consumer
  - ▷ $p \in C \Rightarrow p\cdot x \leqslant p\cdot\omega + p\cdot y$, i.e., $p\cdot z \leqslant 0 \Leftrightarrow$: $p\cdot\zeta(p) \leqslant 0$
  - ▷ free disposal: replace = by $\leqslant$ in $x^\star - y^\star = \omega \Rightarrow \exists$ equilibrium $\Leftrightarrow \exists p \in C$ s.t. $\zeta(p) \cap (-\Omega) \neq \emptyset$
  - ▷ free disposal: $\eta_j(p)$ only def. if $p \geqslant 0 \Rightarrow C \subset \Omega$
  - ▷ $\zeta(p/\sum p_h) = \zeta(p) \Rightarrow$ restrict to $P = \{p \in \Omega : \sum p_h = 1\}$
  - ▷ $Z \subset \mathbb{R}^l$ compact, $\zeta : P \twoheadrightarrow Z$ upper semicts., $\forall p \in P$: $\zeta(p)$ nonempty, convex and $p\cdot\zeta(p) \leqslant 0 \Rightarrow \exists p^\star \in P$ s.t. $\zeta(p^\star) \cap (-\Omega) \neq \emptyset$
- ▶ $\mathcal{E} = ((X_i,\precsim_i),(Y_j),(\omega_i),(\theta_{ij}))$ has an equilibrium if $\forall i$ and $\forall j$:
  - (a) X_i closed, convex, lower bound for $\leqslant$
  - (b.1) no satiation consumption in X_i
  - (b.2) $\forall x'_i \in X_i : \{x_i \in X_i :\ x_i \succsim x'_i\}$ and $\{x_i \in X_i : x'_i \precsim x_i\}$ are closed in X_i
  - (b.3) $x^2_i \succ_i x^1_i \Rightarrow t x^2_i + (1-t) x^1_i \succ_i x^1_i$ for any x^1_i and x^2_i in X_i and $0 < t < 1$.
  - (c) $\exists x^0_i \in X_i$ s.t. $x^0_i \ll \omega_i$
  - (d.1) $0 \in Y_j$
  - (d.2) Y closed and convex
  - (d.3) $Y \cap (-Y) \subset \{0\}$
  - (d.4) $Y \supset (-\Omega)$

6 Optimum

- ▶ preordering on $A : ((x_i),(y_j)) \precsim ((x'_i),(y'_j))$ if $x_i \precsim_i x'_i$ $\forall i$, optimum = maximal element
- ▶ $E = ((X_i,\precsim_i),(Y_j),\omega)$ has an optimum if $\forall i$:
  - (a) X_i closed, connected, lower bound for $\leqslant$
  - (b) $\forall x'_i \in X_i : \{x_i \in X_i :\ x_i \succsim x'_i\}$ and $\{x_i \in X_i : x'_i \precsim x_i\}$ are closed in X_i
  - (c) Y closed, convex, $Y \cap \Omega = \{0\}$
  - (d) $\omega \in X - Y$
- ▶ equilibrium $((x^\star_i),(y^\star_j))$ relative to given p s.t.:
  - (α) $x^\star_i$ greatest element of $\{x_i \in X_i :\ p \cdot x_i \leqslant p \cdot x^\star_i\}$
  - (β) $y^\star_j$ max. profit relative to p on Y_j
  - (γ) $x^\star - y^\star = \omega$
- ▶ $((x^\star_i),(y^\star_j))$ equilibrium relative to $p^\star \Leftrightarrow ((x^\star_i),(y^\star_j),p^\star)$ equilibrium of private ownership economy with $\omega_i = x^\star_i - y^\star/m$, $\theta_{ij} = 1/m$
- ▶ $X^{x^\star_i}_i = \{x_i \in X_i :\ x_i \succsim_i x^\star_i\}$
- ▶ $G = \sum X^{x^\star_i}_i - \sum Y_j$, total resources necessary to improve upon $x^\star_i$
- ▶ X_i convex, preferences convex, $((x^\star_i),(y^\star_j))$ equilibrium relative to p, $x^\star_i$ no satiation $\Rightarrow ((x^\star_i),(y^\star_j))$ optimum
- ▶ $E = ((X_i,\precsim_i),(Y_j))$ s.t. $\forall i$:
  - (a) X_i convex
  - (b.1) $\forall x'_i \in X_i : \{x_i \in X_i :\ x_i \succsim x'_i\}$ and $\{x_i \in X_i : x'_i \precsim x_i\}$ are closed in X_i
  - (b.2) $x^2_i \succ_i x^1_i \Rightarrow t x^2_i + (1-t) x^1_i \succ_i x^1_i$ for any x^1_i and x^2_i in X_i and $0 < t < 1$.
  - (c) Y convex

  given optimum $((x^\star_i),(y^\star_j))$, some $x^\star_{i'}$ not satiation $\Rightarrow \exists p \neq 0$ s.t.:
  - (α) $x^\star_i$ min. $p \cdot x_i$ on $\{x_i \in X_i :\ x_i \succsim_i x^\star_i\}$
  - (β) $y^\star_j$ max. profit relative to p on Y_j
- ▶ if in addition $p\cdot x^\star_i \neq \min p\cdot X_i$, then above equilibrium relative to p

7 Uncertainty

- ▶ incorporate uncertainty by extending commodity space: replace "date" by "event" for commodities and prices

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