SUGGESTED ANSWERS FOR MACRO COMPS - 2000-2004

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WARNING: The answers suggested here are a collection of old stuffs. I am sure you will find a lot of mistakes and typos. When you think my answers are wrong just skip them and stick to your own solutions. I'm also sure you are able to find more neat solutions than the ones suggested here. Try to discuss the Q&A with your classmates, and with senior students. Please let me know all mistakes and typos you will certainly find.

FALL 2000 - QUESTION 2

(a) Planner's problem:

$$\max \sum_{t=0}^{\infty} \beta^t \{\lambda[\log C_{1t} + B\log(1-N_{1t})] + (1-\lambda)[\log C_{2t} + B\log(1-N_{2t})]\}$$

$$\text{s.t. } C_{1t} + C_{2t} = A_1^{1-\theta}N_{1t}^{1-\theta} + A_2^{1-\theta}N_{2t}^{1-\theta}$$

where λ is the weight of individual and $1-\lambda$ is the weight of individual 2. Since the planner cares equally about the individuals we can set $\lambda = 1/2$.

Let η_t be the lagrange multiplier on the resource constraint at time t. The FOCs are:

$$C_{1t} : \frac{1/2}{C_{1t}} = \eta_t$$
$$C_{2t} : \frac{1/2}{C_{2t}} = \eta_t$$
$$N_{1t} : \frac{1/2B}{1-N_{1t}} = \eta_t \frac{A_1^{1-\theta}N_{1t}^{1-\theta}}{N_{1t}}$$
$$N_{2t} : \frac{1/2B}{1-N_{2t}} = \eta_t \frac{A_2^{1-\theta}N_{2t}^{1-\theta}}{N_{2t}}$$

After combining these conditions and rearranging terms we get:

$$\frac{C_{1t}}{C_{2t}} = 1 \Rightarrow C_{1t} = C_{2t} \equiv C_t$$
$$\frac{BN_{1t}}{1-N_{1t}} = \frac{A_1^{1-\theta}N_{1t}^{1-\theta}}{C_t}$$
$$\frac{BN_{2t}}{1-N_{2t}} = \frac{A_2^{1-\theta}N_{2t}^{1-\theta}}{C_t}$$

Solving these last two for N_{1t} and N_{1t} gives:

$$\frac{1-N_{1t}}{N_{1t}^{\theta}} = \frac{BC_t}{A_1^{1-\theta}}$$
$$\frac{1-N_{2t}}{N_{2t}^{\theta}} = \frac{BC_t}{A_2^{1-\theta}}$$

Since $A_1 > A_2$ individual 1 works more than individual 2 because he is more productive. Both individuals consume the same amount because they are equally weighted by the planner. Finally, the problem is static, rather than dynamic, because there is no recursiveness.

(b) Planner's problem:

$$\max \sum_{t=0}^{\infty} \beta^t \left\{ \frac{1}{2}[\alpha C_{1t}^{\sigma} + (1-\alpha)(1-N_{1t})^{\sigma}]^{\frac{1}{\sigma}} + \frac{1}{2}[\alpha C_{2t}^{\sigma} + (1-\alpha)(1-N_{2t})^{\sigma}]^{\frac{1}{\sigma}} \right\}$$

$$\text{s.t. } C_{1t} + C_{2t} = A_1^{1-\theta}N_{1t}^{1-\theta} + A_2^{1-\theta}N_{2t}^{1-\theta}$$

Let η_t be the lagrange multiplier on the resource constraint at time t. The FOCs are:

$$C_{1t} : 1/2\alpha[\alpha C_{1t}^{\sigma} + (1-\alpha)(1-N_{1t})^{\sigma}]^{\frac{1}{\sigma}-1} C_{1t}^{\sigma-1} = \eta_t$$
$$C_{2t} : 1/2\alpha[\alpha C_{2t}^{\sigma} + (1-\alpha)(1-N_{2t})^{\sigma}]^{\frac{1}{\sigma}-1} C_{2t}^{\sigma-1} = \eta_t$$
$$N_{1t} : 1/2(1-\alpha)[\alpha C_{1t}^{\sigma} + (1-\alpha)(1-N_{1t})^{\sigma}]^{\frac{1}{\sigma}-1}(1-N_{1t})^{\sigma-1} = \eta_t \frac{A_1^{1-\theta}N_{1t}^{1-\theta}}{N_{1t}}$$
$$N_{2t} : 1/2(1-\alpha)[\alpha C_{2t}^{\sigma} + (1-\alpha)(1-N_{2t})^{\sigma}]^{\frac{1}{\sigma}-1}(1-N_{2t})^{\sigma-1} = \eta_t \frac{A_2^{1-\theta}N_{2t}^{1-\theta}}{N_{2t}}$$

After combining these conditions and rearranging terms we get:

$$\frac{C_{1t}}{C_{2t}} = \left(\frac{\alpha C_{1t}^{\sigma} + (1-\alpha)(1-N_{1t})^{\sigma}}{\alpha C_{2t}^{\sigma} + (1-\alpha)(1-N_{2t})^{\sigma}} \right)^{\frac{1}{\sigma}}$$
$$\frac{(1-\alpha)N_{1t}}{(1-N_{1t})^{1-\sigma}} = \frac{\alpha A_1^{1-\theta}N_{1t}^{1-\theta}}{C_{1t}^{1-\sigma}}$$
$$\frac{(1-\alpha)N_{2t}}{(1-N_{2t})^{1-\sigma}} = \frac{\alpha A_2^{1-\theta}N_{2t}^{1-\theta}}{C_{2t}^{1-\sigma}}$$

Because preferences are no longer separable in consumption and labor, $C_{1t} > C_{2t}$ and $N_{1t} > N_{2t}$. Note that the above conditions collapse to the conditions found in item (a) when $\sigma \to 0$.

FALL 2000 - QUESTION 3

(a) Note: There is a typo in the question. The model is non stochastic.

Planner's DPP:

$$
\begin{aligned}
v(k,h,A) &= \max_{k',n,s}\{\alpha\log c + (1-\alpha)\log(1-n-s) + \beta v(k',h',A')\} \\
\text{s.t. } c + i &= Ak^{\theta}(nh)^{1-\theta} \\
k' &= (1-\delta_k)k + i \\
h' &= (1-\delta_h)h + s \\
A' &= \gamma A
\end{aligned}
$$

This problem can be simplified to:

$$
v(k,h,A) = \max_{k',n,s}\left\{\alpha\log\left[Ak^{\theta}(nh)^{1-\theta} + (1-\delta_k)k - k'\right] + (1-\alpha)\log(1-n-s) + \beta v(k',(1-\delta_h)h + s, \gamma A)\right\}
$$

(b) Because the economy has two reproducible factors of production (physical capital and human capital), the variables will potenttialy grow at different rates along the BGP. Let g_c be the growth rate of consumption, g_k be the growth rate of physical capital, and g_h be the growth rate of human capital. The variables n and s are already stationary. Using the resource constraint we get:

$$
\begin{aligned}
\frac{c_t}{k_t} + \frac{k_{t+1}}{k_t} &= A_t k_t^{\theta-1}(n_t h_t)^{1-\theta} + 1 - \delta_k \\
\left(\frac{g_c}{g_k}\right)^t \frac{\hat{c}_t}{\hat{k}_t} + g_k &= \left(\gamma\frac{g_h^{1-\theta}}{g_k^{1-\theta}}\right)^t \hat{k}_t^{\theta-1}\left(n_t\hat{h}_t\right)^{1-\theta} + 1 - \delta_k
\end{aligned}
$$

where the hat denotes stationary variables.

The left-hand side of the above expression is stationary if $g_c = g_k \equiv g$. The right-hand side is stationary if

$g_k^{1-\theta} = \gamma g_h^{1-\theta}$. This implies that

$$
g_c = g_k \equiv g = \gamma^{\frac{1}{1-\theta}} g_h
$$

Now use the law of motion of h to pin down g_h:

$$
\begin{aligned}
\frac{h_{t+1}}{h_t} &= 1 - \delta_h + \frac{s_t}{h_t} \\
g_h &= 1 - \delta_h + \frac{s_t}{g_h^t \hat{h}_t}
\end{aligned}
$$

This expression is stationary if

$$
g_h = 1 \tag{1}
$$

Therefore,

$$
g = \gamma^{\frac{1}{1-\theta}} \tag{2}
$$

Characterization of the BGP:

$$\begin{aligned} c_t &= \gamma^{\frac{1}{1-\theta}} \hat{c}_t \\ k_t &= \gamma^{\frac{1}{1-\theta}} \hat{k}_t \\ y_t &= \gamma^{\frac{1}{1-\theta}} \hat{y}_t \\ h_t &= \hat{h}_t \end{aligned}$$

(c) First, write down the planner's stationary problem:

$$v(\hat{k}, \hat{h}) = \max_{\hat{k}', n, s} \left\{ \alpha \log\left[\hat{k}^{\theta} \left(n\hat{h} \right)^{1-\theta} + (1-\delta_k)\hat{k} - \gamma^{\frac{1}{1-\theta}} \hat{k}' \right] + (1-\alpha)\log(1-n-s) + \beta v(\hat{k}', (1-\delta_h)\hat{h} + s) \right\}$$

Second, derive the stationary optimality conditions:

$$\text{FOC } \hat{k}' : \frac{\alpha \gamma^{\frac{1}{1-\theta}}}{\hat{c}} = \beta v_1(\hat{k}', \hat{h}') \tag{3.1}$$

$$\text{FOC } n : \frac{\alpha(1-\theta) k^{\theta} \left(n\hat{h} \right)^{1-\theta}}{\hat{c} n} = \frac{1-\alpha}{1-n-s} \tag{3.2}$$

$$\text{FOC } s : \frac{1-\alpha}{1-n-s} = \beta v_2(\hat{k}', \hat{h}') \tag{3.3}$$

$$\text{EC } \hat{k} : v_1(\hat{k}, \hat{h}) = \frac{\alpha}{\hat{c}} \left(\frac{\theta \hat{k}^{\theta} \left(n\hat{h} \right)^{1-\theta}}{\hat{k}} + 1 - \delta_k \right) \tag{3.4}$$

$$\text{EC } \hat{h} : v_2(\hat{k}, \hat{h}) = \frac{\alpha(1-\theta)\hat{k}^{\theta} \left(n\hat{h} \right)^{1-\theta}}{\hat{c}\hat{h}} + \beta v_2(\hat{k}', \hat{h}')(1-\delta_h) \tag{3.5}$$

Substituting (3.4) into 3.(1) gives:

$$\frac{\gamma^{\frac{1}{1-\theta}} \hat{c}'}{\hat{c}} = \beta \left(\frac{\theta \hat{k}'^{\theta} \left(n'\hat{h}' \right)^{1-\theta}}{\hat{k}'} + 1 - \delta_k \right) \tag{3.6}$$

Substituting (3.3) into (3.5) gives:

$$v_2(\hat{k}, \hat{h}) = \frac{\alpha(1-\theta)\hat{k} \left(n\hat{h} \right)^{1-\theta}}{\hat{c}\hat{h}} + \frac{1-\alpha}{1-n-s}(1-\delta_h)$$

Therefore,

$$v_2(\hat{k}', \hat{h}') = \frac{\alpha(1-\theta)\hat{k}'^{\theta} \left(n'\hat{h}' \right)^{1-\theta}}{\hat{c}'\hat{h}'} + \beta \frac{1-\alpha}{1-n'-s'}(1-\delta_h)$$

Substituting this back into (3.3) gives:

$$\frac{1-\alpha}{1-n-s} = \frac{\alpha(1-\theta)\hat{k}'^{\theta} \left(n'\hat{h}' \right)^{1-\theta}}{\hat{c}'\hat{h}'} + \beta \frac{1-\alpha}{1-n'-s'}(1-\delta_h) \tag{3.7}$$

Collecting equations (3.2), (3.6), (3.7), and the other constraints we get:

$$\begin{aligned}
\text{Production function} &: \hat{y}_t = \hat{k}_t^{\theta}\left(n_t\hat{h}_t\right)^{1-\theta} \\
\text{Resource constraint} &: \hat{c}_t = \hat{y}_t + \hat{\imath}_t \\
\text{Law of motion of capital} &: \hat{k}_{t+1} = (1-\delta_k)\hat{k}_t + \hat{\imath}_t \\
\text{Law of motion of human capital} &: \hat{h}_{t+1} = (1-\delta_h)\hat{h}_t + s_t \\
\text{Labor choice} &: \frac{\alpha(1-\theta)\hat{y}_t}{\hat{c}_t n_t} = \frac{1-\alpha}{1-n_t-s_t} \\
\text{Physical capital euler equation} &: \frac{\gamma^{\frac{1}{1-\theta}}\hat{c}_{t+1}}{\hat{c}_t} = \beta\left(\frac{\theta\hat{y}_{t+1}}{\hat{k}_{t+1}} + 1 - \delta_k\right) \\
\text{Human capital euler equation} &: \frac{1-\alpha}{1-n_t-s_t} = \beta\left(\frac{\alpha(1-\theta)\hat{y}_{t+1}}{\hat{c}_{t+1}\hat{h}_{t+1}} + \frac{(1-\alpha)(1-\delta_h)}{1-n_{t+1}-s_{t+1}}\right)
\end{aligned}$$

Third, get the steady state system of equations:

$$\hat{y} = \hat{k}^{\theta}\left(n\hat{h}\right)^{1-\theta} \tag{4.1}$$

$$\hat{c} = \hat{y} + \hat{\imath} \tag{4.2}$$

$$\hat{\imath} = \delta_k\hat{k} \tag{4.3}$$

$$s = \delta_h\hat{h} \tag{4.4}$$

$$\frac{\alpha(1-\theta)}{n}\frac{\hat{y}}{\hat{c}} = \frac{1-\alpha}{1-n-s} \tag{4.5}$$

$$\gamma^{\frac{1}{1-\theta}} = \beta\left(\theta\frac{\hat{y}}{\hat{k}} + 1 - \delta_k\right) \tag{4.6}$$

$$\frac{1-\alpha}{1-n-s} = \beta\left(\frac{\alpha(1-\theta)}{\hat{h}}\frac{\hat{y}}{\hat{c}} + \frac{(1-\alpha)(1-\delta_h)}{1-n-s}\right) \tag{4.7}$$

Calibration strategy:

Parameter to be calibrated	Data Moment Needed	Equation used to calibrate
α	fraction of hours worked	4.5
β	capital-output ratio	4.6
γ	GDP growth rate	
δ_k	investment-output ratio	4.3
δ_h	fraction of time spent in education	4.4
θ	fraction of capital income on total income	

(d) Household's DPP:

$$v(k,h,b,K,H,A) = \max_{k',b',n,s}\{\alpha\log c + (1-\alpha)\log(1-n-s) + \beta v(k',h',b',K',H,A')\} \quad (5)$$

$$\begin{aligned} \text{s.t. } c + k' + q(K,H,A)b' &= r(K,H,A)k + w(K,H,A)(nh) + (1-\delta_k)k + b \\ h' &= (1-\delta_h)h + s \\ K' &= G_1(K,H,A) \\ H' &= G_2(K,H,A) \\ A' &= A + \varepsilon' \end{aligned}$$

where b is the one-period contingent claim and $q(K,A)$ is the contingent discount price.

Final goods producer:

$$\max_{K,L}\left\{A(K)^{\theta}(L)^{1-\theta} - r(K,H,A)K - w(K,H,A)L\right\} \quad (6)$$

Note: I am using capital letters to denote firm's choices.

Definition: A recursive competitive equilibrium for this economy is:

(i) A set of decision rules $k'(k,h,b,K,H,A), b'(k,h,K,H,A), n(k,h,b,K,H,A), s(k,h,b,K,H,A)$

(ii) A set of decision rules $K(K,H,A), L(K,H,A)$ for the firm

(iii) Pricing functions $r(K,H,A), w(K,H,A)$

(iv) Aggregate lawS of motion $K' = G_1(K,H,A)$ and $H' = G_2(K,H,A)$

such that

(1) Given (iii) and (iv), (i) solves problem (5)

(2) Given (iii), (ii) solves problem (6)

(3) Markets clear:

$$\begin{aligned} L &= n(K,H,0,K,H,A)h(K,H,0,K,H,A) \\ K &= k(K,H,0,K,H,A) \\ 0 &= b'(k,h,K,H,A) \end{aligned}$$

By Walras' law the output market also clears.

(4) Perceptions are correct:

$$\begin{aligned} k'(K,H,0,K,H,A) &= G_1(K,H,A) \\ (1-\delta_h)H - s(K,H,0,K,H,A) &= G_2(K,H,A) \end{aligned}$$

(e) A positive technology shock raises the marginal product of effective labor ("effective labor" $= nh$), hence it also raises the real wage rate per units of effective labor. The amount of raw labor (n) will increase at the impact. Since the return on education also increases, s will also increase. If my claimi is right, s is procyclical. Be careful, however, because the technology shock is a random walk, which means the shocks are permanent. Therefore, a positive technology shock is equivalent to a permanent income shock. Try to figure out the effects of a permanent increase in income and check if my claim is right.

SPRING 2000 -QUESTION 2

(a) They are consistent with balanced growth path. In particular,

(i) the utility function is consistent with the empirical fact of an elasticity of substitution between consumption and leisure close to 1

(ii) the production function is consistent with the empirical observation of constant input shares

(b) To make the discussion clear, note that the steady state equations that characterize the planner's solution are:

$$\text{Labor choice}: \frac{A\bar{h}}{1-\bar{h}} = (1-\theta)\frac{\bar{y}}{\bar{c}}$$

$$\text{Euler equation}: 1 = \beta\left(\frac{\theta\bar{y}}{\bar{k}} + 1 - \delta\right)$$

$$\text{Law of motion of capital}: \bar{\imath} = \delta\bar{k}$$

$$\text{Resource constraint}: \bar{c} + \bar{\imath} = \bar{y}$$

Calibration strategy:

Parameter to be calibrated	Data Moment Needed	Equation used to calibrate
A	fraction of hours worked	Labor choice
β	capital-output ratio	Euler equation
δ	investment-output ratio	Law of motion of capital
θ	fraction of capital income on total income	

As for the markov chain, assume that

$$P = \begin{pmatrix} p_1 & 1-p_1 \\ 1-p_2 & p_2 \end{pmatrix}$$

where p_1 is the probability of the low state (recession) and p_2 is the probability of high state (boom).

Note that if $p_1 >$ and $p_2 > 0$, the invariant distribution is unique and equal to

$$\left(\frac{1-p_2}{2-p_1-p_2}, \frac{1-p_1}{2-p_1-p_2}\right)$$

In order to calibrate these two probabilities one could look at the times series of GDP during recessions and booms and then estimate the persistence of each state. The probabilities p_1 and p_2 would them have to match the persistence of recessions and booms, respectively.

Another way would be to approximate the solow residual z_t by an AR(1) process, such as $z_{t+1} = (1-\rho)\mu + \rho z_t + \varepsilon_{t+1}$, where ε is i.i.d. with zero mean and variance σ^2. One could fit this equation by OLS or other estimation method and come up with estimates for ρ, μ and σ. Then one could approximate this AR(1) process by the two state markov chain above. Use the conditional and uncondintional first and second moments to calibrate (z_H, z_L) and (p_1, p_2):

$$\begin{aligned}
E[z'|z = z_L] &= z_L p_1 + (1-p_1) z_H = (1-\rho)\mu + \rho z_L \\
E[z'|z = z_H] &= z_H p_2 + (1-p_2) z_L = (1-\rho)\mu + \rho z_H \\
E[z] &= z_L \frac{1-p_2}{2-p_1-p_2} + z_H \frac{1-p_1}{2-p_1-p_2} = \mu \\
Var(z) &= z_L^2 \frac{1-p_2}{2-p_1-p_2} + z_H^2 \frac{1-p_1}{2-p_1-p_2} = \frac{\sigma^2}{1-\rho^2}
\end{aligned}$$

(c), (d) See Hansen's HW4.

(e) It was shown in Hansen's HW4 that the model with indivisible labor, lotteries, and log utility implies the following type of preferences: $u(c, 1-h) = \log c - B\pi\hat{h}$, where $B = -A\log(1-\hat{h})/\hat{h}$ and π is the probability of working.

Household's DPP:

$$\begin{aligned}
v(K,k,b,z) &= \max_{k',b',\pi} \left\{ \log c - B\pi\hat{h} + \beta E[v(K',k',b',z')] \right\} \qquad (1) \\
c &= r(K,z)k + w(K,z)\pi\hat{h} + (1-\delta)k + b - k' - q(K,z)b' \\
K' &= G(K,z) \\
z' &\in \{z_L, z_H\} \text{ with transition matrix } P
\end{aligned}$$

where $q(K,z)$ is the contingent price of the one-period contingent bond b.

Firm's Problem:

$$\max_{k^f,h^f} \left\{ e^z (k^f)^\theta (h^f)^{1-\theta} - r(K,z)k^f - w(K,z)h^f \right\} \qquad (2)$$

Definition: A RCE for this economy is:

(1) A set of decision rules $k'(K,k,b,z), b'(K,k,b,z)$ and $\pi(K,k,b,z)$ for the household.

(2) A set of decision rules $k^f(K,z)$ and $h^f(K,z)$ for the firm.

(3) A set of pricing functions $r(K,z)$, $w(K,z)$ and $q(K,z)$.

(4) An aggregate law of motion for capital $G(K,z)$.

such that

(i) Given (3) and (4), (1) solves the household's DPP.

(ii) Given (3), (2) solves the firm's problem.

(iii) Markets clear:

$$
\begin{aligned}
k^f(K,z) &= K \\
h^f(K,z) &= \pi(K,K,0,z)\hat{h} \\
b(K,K,0,z) &= 0
\end{aligned}
$$

(iv) Perceptions are correct:

$$k'(K,K,0,z) = G(K,z)$$

SPRING 2000 -QUESTION 3

(a) Planner's DPP:

$$v(K_1,K_2,A_1,A_2) = \max_{N_1,N_2,K_1',K_2'} \left\{ \begin{array}{c} [\alpha(A_1K_1^{\theta}N_1^{1-\theta} - K_2')^{\sigma} + (1-\alpha)(A_2K_2^{\theta}N_2^{1-\theta} - K_1')^{\sigma}]^{\frac{1}{\sigma}} \\ -B(N_1+N_2) + \beta E[v(K_1',K_2',A_1',A_2')] \end{array} \right\}$$

$$
\begin{aligned}
A_1' &= 1+\varepsilon_1' \\
A_2' &= 1+\varepsilon_2'
\end{aligned}
$$

FOCs:

$$
\begin{aligned}
N_1 &: [\alpha C_1^{\sigma} + (1-\alpha)C_2^{\sigma}]^{\frac{1}{\sigma}-1}\alpha C_1^{\sigma-1}(1-\theta)A_1K_1^{\theta}N_1^{-\theta} = B \\
N_2 &: [\alpha C_1^{\sigma} + (1-\alpha)C_2^{\sigma}]^{\frac{1}{\sigma}-1}(1-\alpha)C_2^{\sigma-1}(1-\theta)A_2K_2^{\theta}N_2^{-\theta} = B \\
K_1' &: [\alpha C_1^{\sigma} + (1-\alpha)C_2^{\sigma}]^{\frac{1}{\sigma}-1}(1-\alpha)C_2^{\sigma-1} = \beta E[v_1(K_1',K_2',A_1',A_2')] \\
K_2' &: [\alpha C_1^{\sigma} + (1-\alpha)C_2^{\sigma}]^{\frac{1}{\sigma}-1}\alpha C_1^{\sigma-1} = \beta E[v_2(K_1',K_2',A_1',A_2')]
\end{aligned}
$$

ECs:

$$
\begin{aligned}
K_1 &: v_1(K_1,K_2,A_1,A_2) = [\alpha C_1^{\sigma} + (1-\alpha)C_2^{\sigma}]^{\frac{1}{\sigma}-1}\alpha C_1^{\sigma-1}\theta A_1K_1^{\theta-1}N_1^{1-\theta} \\
K_2 &: v_2(K_1,K_2,A_1,A_2) = [\alpha C_1^{\sigma} + (1-\alpha)C_2^{\sigma}]^{\frac{1}{\sigma}-1}(1-\alpha)C_2^{\sigma-1}\theta A_2K_2^{\theta-1}N_2^{1-\theta}
\end{aligned}
$$

Substituting the ECs into the two last FOCs gives the following two euler equations:

$$
\begin{aligned}
(1-\alpha)[\alpha C_1^{\sigma} + (1-\alpha)C_2^{\sigma}]^{\frac{1}{\sigma}-1}C_1^{\sigma-1} &= \beta E\left(\alpha C_1'^{\sigma-1}[\alpha C_1'^{\sigma} + (1-\alpha)C_2'^{\sigma}]^{\frac{1}{\sigma}-1}\theta A_1'K_1'^{\theta-1}N_1'^{1-\theta}\right) \\
\alpha[\alpha C_1^{\sigma} + (1-\alpha)C_2^{\sigma}]^{\frac{1}{\sigma}-1}C_2^{\sigma-1} &= \beta E\left((1-\alpha)C_2'^{\sigma-1}[\alpha C_1'^{\sigma} + (1-\alpha)C_2'^{\sigma}]^{\frac{1}{\sigma}-1}\theta A_2'K_2'^{\theta-1}N_2'^{1-\theta}\right)
\end{aligned}
$$

(b) The steady state system of equations is:

$$
\begin{aligned}
[\alpha\bar{C}_1^{\sigma} + (1-\alpha)\bar{C}_2^{\sigma}]^{\frac{1}{\sigma}-1}\alpha\bar{C}_1^{\sigma-1}(1-\theta)\bar{K}_1^{\theta}\bar{N}_1^{-\theta} &= B \\
[\alpha\bar{C}_1^{\sigma} + (1-\alpha)\bar{C}_2^{\sigma}]^{\frac{1}{\sigma}-1}(1-\alpha)\bar{C}_2^{\sigma-1}(1-\theta)\bar{K}_2^{\theta}\bar{N}_2^{-\theta} &= B \\
\frac{1-\alpha}{\alpha} &= \beta\theta\bar{K}_1^{\theta-1}\bar{N}_1^{1-\theta} \\
\frac{\alpha}{1-\alpha} &= \beta\theta\bar{K}_2^{\theta-1}\bar{N}_2^{1-\theta} \\
\bar{C}_1 &= \bar{K}_1^{\theta}\bar{N}_1^{1-\theta} - \bar{K}_1 \\
\bar{C}_2 &= \bar{K}_2^{\theta}\bar{N}_2^{1-\theta} - \bar{K}_2
\end{aligned}
$$

Check: 6 equations and 6 unknowns $(\bar{C}_1,\bar{C}_2,\bar{N}_1,\bar{N}_2,\bar{K}_1,\bar{K}_2)$.

(c) Household's DPP:

$$
v(k_1,k_2,K_1,K_2,A_1,A_2) = \max_{n,k_1',k_2',C_1,C_2}\left\{\begin{array}{c} [\alpha C_1^{\sigma} + (1-\alpha)C_2^{\sigma}]^{\frac{1}{\sigma}} - Bn + \\ \beta E[v(k_1',k_2',K_1',K_2',A_1',A_2')] \end{array}\right\}
$$

$$
\begin{aligned}
C_1 + k_2' + p(K_1,K_2,A_1,A_2)(C_2 + k_1') &= w(K_1,K_2,A_1,A_2)n + r(K_1,K_2,A_1,A_2)(k_1 + k_2) \\
K_1' &= G_1(K_1,K_2,A_1,A_2) \\
K_2' &= G_2(K_1,K_2,A_1,A_2) \\
A_1' &= 1 + \varepsilon_1' \\
A_2' &= 1 + \varepsilon_2'
\end{aligned}
$$

Remarks: (i) p is the relative price of the final good 2, (ii) I am assuming perfect factor mobility.

Firm 1's problem:

$$
\max_{K_1,N_1}\left\{A_1\left(K_1^f\right)^{\theta}\left(N_1^f\right)^{1-\theta} - r(K_1,K_2,A_1,A_2)K_1^f - w(K_1,K_2,A_1,A_2)N_1^f\right\}
$$

Firm 2's problem:

$$
\max_{K_2,N_2}\left\{p(K_1,K_2,A_1,A_2)A_2\left(K_2^f\right)^{\theta}\left(N_2^f\right)^{1-\theta} - r(K_1,K_2,A_1,A_2)K_2^f - w(K_1,K_2,A_1,A_2)N_2^f\right\}
$$

Definition: A recursive competitive equilibrium for this economy is:

(i) A set of decision rules $k_1'(k_1,k_2,K_1,K_2,A_1,A_2)$, $k_2'(k_1,k_2,K_1,K_2,A_1,A_2)$, $C_1(k_1,k_2,K_1,K_2,A_1,A_2)$, $C_2(k_1,k_2,K_1,K_2,A_1,A_2)$, $n(k_1,k_2,K_1,K_2,A_1,A_2)$ for the household.

(ii) A set of decision rules $K_1^f(K_1,K_2,A_1,A_2), N_1^f(K_1,K_2,A_1,A_2)$ for firm 1.

(iii) A set of decision rules $K_2^f(K_1,K_2,A_1,A_2), N_2^f(K_1,K_2,A_1,A_2)$ for firm 2.

(iv) Pricing functions $r(K_1,K_2,A_1,A_2), w(K_1,K_2,A_1,A_2)$ and $p(K_1,K_2,A_1,A_2)$.

(v) Aggregate laws of motion $G_1(K_1,K_2,A_1,A_2), G_2(K_1,K_2,A_1,A_2)$

such that

(1) Given (iv) and (v), (i) solves the household problem.

(2) Given (iv), (ii) solves the problem of firm 1.

(3) Given (iv), (iii) solves the problem of firm 2.

(4) Markets clear:

$$
\begin{aligned}
K_1^f(K_1,K_2,A_1,A_2) &= k_1(K_1,K_2,K_1,K_2,A_1,A_2)\\
K_2^f(K_1,K_2,A_1,A_2) &= k_2(K_1,K_2,K_1,K_2,A_1,A_2)\\
N_1^f(K_1,K_2,A_1,A_2)+N_2^f(K_1,K_2,A_1,A_2) &= n(K_1,K_2,K_1,K_2,A_1,A_2)\\
C_1(K_1,K_2,K_1,K_2,A_1,A_2) &= A_1(K_1)^\theta(N_1)^{1-\theta}-k_1'(K_1,K_2,K_1,K_2,A_1,A_2)\\
C_2(K_1,K_2,K_1,K_2,A_1,A_2) &= A_2(K_2)^\theta(N_2)^{1-\theta}-k_2'(K_1,K_2,K_1,K_2,A_1,A_2)
\end{aligned}
$$

(5) Perceptions are correct:

$$
\begin{aligned}
k_1'(K_1,K_2,K_1,K_2,A_1,A_2) &= G_1(K_1,K_2,A_1,A_2)\\
k_2'(K_1,K_2,K_1,K_2,A_1,A_2) &= G_2(K_1,K_2,A_1,A_2)
\end{aligned}
$$

The planner solution coincides with the market solution because both welfare theorem hold.

(d) Let $\bar{p}$ be the relative price of sector 2 output:

$$
\begin{aligned}
Y_t &= Y_{1t}+\bar{p}Y_{2t}\\
&= A_{1t}(K_{1t})^\theta(N_{1t})^{1-\theta}+\bar{p}A_{2t}(K_{2t})^\theta(N_{2t})^{1-\theta}\\
\Rightarrow\ \frac{\partial\log Y_t}{\partial\log A_{1t}} &= (K_{1t})^\theta(N_{1t})^{1-\theta} = \frac{Y_{1t}}{A_{1t}} > 0
\end{aligned}
$$

If the above computation is right, the macroeconomic impact of sectoral shocks does not depend on α and σ. Try to check my solution because this result is quite counterintuitive in my opinion.

FALL 2001 - QUESTION 2

(a)

(i) Planner's DPP:

$$
v(k,z) = \max_{k',n,c_1,c_2}\left\{nU(c_1,1-\hat{h})+(1-n)U(c_2,1)+\beta E[v(k',z')]\right\}
$$

$$
\text{s.t. } nc_1+(1-n)c_2 = zF(k,n\hat{h})+(1-\delta)k-k'
$$

$$
z'\sim G(z',z)
$$

where n is the probability of working (or the fraction of people working), c_1 is consumption when employed and c_2 is consumption when unemployed.

(ii) System of equations that characterize the solution:

$$\text{FOCs } c_1, c_2 \;:\; U_1(c_1, 1-\hat{h}) = U_1(c_2, 1)$$

$$\text{FOC } n \;:\; U(c_2, 1) - U(c_1, 1-\hat{h}) = U_1(c_1, 1-\hat{h})\left(zF_2(k, n\hat{h}) - c_1 + c_2\right)$$

$$\text{Euler equation } \;:\; U_1(c_1, 1-\hat{h}) = \beta\left[U_1(c_1', 1-\hat{h})\left(z'F_1(k', n'\hat{h}) + 1 - \delta\right)\right]$$

$$\text{Resource constraint } : nc_1 + (1-n)c_2 = zF(k, n\hat{h}) + (1-\delta)k - k'$$

(b)

(i) Planner's DPP:

$$v(k, k', \ldots, k^{J-1}, z) = \max_{k^J, h}\{U(c, 1-h) + \beta E[v(k', k'', \ldots, k^J, z')]\}$$

$$\text{s.t. } c = zF(k,h) - i$$

$$i = \frac{1}{J}P + \frac{1}{J}P_{-1} + \ldots + \frac{1}{J}P_{-J+1} = \frac{1}{J}(k' + k'' + \ldots + k^J) - \frac{1}{J}(1-\delta)(k + k' + \ldots + k^{J-1})$$

$$k_J = (1-\delta)k_{J-1} + P$$

$$z' \sim G(z', z)$$

(ii) System of equations that characterize the solution:

$$\text{FOC } h \;:\; U_2(c, 1-h) = U_1(c, 1-h)zF_2(k,h)$$

Euler equation :

$$U_1(c, 1-h)\frac{1}{J} = \beta E\left[\begin{array}{c} [U_1(c', 1-h') + \beta E U_1(c'', 1-h'') + \ldots + \beta^{J-2}EU_1(c^{J-1}, 1-h^{J-1})](-\tfrac{1}{J}\delta) \\ +\beta[\beta^{J-1}EU_1(c^J, 1-h^J)(z^J F_1(k^J, h^J) + \tfrac{1}{J}(1-\delta))] \end{array}\right]$$

$$\text{Resource contraint } : c = zF(k,h) + \frac{1}{J}(1-\delta)(k + k' + \ldots + k^{J-1}) - \frac{1}{J}(k' + k'' + \ldots + k^J)$$

FALL 2001 - QUESTION 4

(a) The consumer is indifferent between the two consumption streams if

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\alpha} - 1}{1-\alpha} = \sum_{t=0}^{\infty} \beta^t \frac{\bar{c}_t^{1-\alpha} - 1}{1-\alpha} \tag{1}$$

Substituting for c_t gives:

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\alpha} - 1}{1-\alpha} = E_0 \sum_{t=0}^{\infty} \beta^t \frac{\left[(1+\Omega)\bar{c}_t e^{z_t - \frac{1}{2}\sigma_z^2}\right]^{1-\alpha} - 1}{1-\alpha}$$

$$= \sum_{t=0}^{\infty} \beta^t \frac{[(1+\Omega)\bar{c}_t]^{1-\alpha} e^{-(1-\alpha)\frac{1}{2}\sigma_z^2} E_0(e^{(1-\alpha)z_t}) - 1}{1-\alpha}$$

Given that z is i.i.d normally distributed, then $e^{(1-\alpha)z}$ is i.i.d lognormally distributed with mean $e^{(1-\alpha)^2\frac{1}{2}\sigma_z^2}$ (see Casella and Berger for the relationship between the normal and lognormal distributions). Therefore,

$$
\begin{aligned}
E_0\sum_{t=0}^{\infty}\beta^t\frac{c_t^{1-\alpha}-1}{1-\alpha} &= \sum_{t=0}^{\infty}\beta^t\frac{[(1+\Omega)\bar{c}_t]^{1-\alpha}e^{-(1-\alpha)\frac{1}{2}\sigma_z^2}e^{(1-\alpha)^2\frac{1}{2}\sigma_z^2}-1}{1-\alpha} \\
&= \sum_{t=0}^{\infty}\beta^t\frac{[(1+\Omega)\bar{c}_t]^{1-\alpha}e^{-\alpha(1-\alpha)\frac{1}{2}\sigma_z^2}-1}{1-\alpha} \\
&= (1+\Omega)^{1-\alpha}e^{-\alpha(1-\alpha)\frac{1}{2}\sigma_z^2}\sum_{t=0}^{\infty}\beta^t\frac{\bar{c}_t^{1-\alpha}}{1-\alpha}-\frac{1}{(1-\alpha)(1-\beta)} \qquad (2)
\end{aligned}
$$

Substitute (2) into (1) to get:

$$
\begin{aligned}
(1+\Omega)^{1-\alpha}e^{-\alpha(1-\alpha)\frac{1}{2}\sigma_z^2}\sum_{t=0}^{\infty}\beta^t\frac{\bar{c}_t^{1-\alpha}}{1-\alpha}-\frac{1}{(1-\alpha)(1-\beta)} &= \sum_{t=0}^{\infty}\beta^t\frac{\bar{c}_t^{1-\alpha}-1}{1-\alpha} \\
(1+\Omega)^{1-\alpha}e^{-\alpha(1-\alpha)\frac{1}{2}\sigma_z^2}\sum_{t=0}^{\infty}\beta^t\frac{\bar{c}_t^{1-\alpha}}{1-\alpha} &= \sum_{t=0}^{\infty}\beta^t\frac{\bar{c}_t^{1-\alpha}}{1-\alpha} \\
(1+\Omega)^{1-\alpha} &= \left(e^{\frac{\alpha}{2}\sigma_z^2}\right)^{1-\alpha} \\
1+\Omega &= e^{\frac{\alpha}{2}\sigma_z^2} \\
\Omega &= \exp\left(\frac{\alpha\sigma_z^2}{2}\right)-1
\end{aligned}
$$

The coefficient of relative risk aversion α is usually estimated around 2. Additionally, assuming that the stochastic process z_t probably describes business cycles fluctuations, we have that estimates of σ_z for the US are small than 1 percent at a quarterly basis. Therefore, the value of $\exp(\alpha\sigma_z^2/2)$ is only a little bit larger than 1. This implies that Ω is very close to zero. The parameter Ω is the additional consumption required by the agent to compensate for fluctuations in his consumption stream because of business cycle volatility. Therefore, it is a measure of welfare cost of business cycles. Since we concluded that Ω is very close to zero for reasonable values of α and σ_z then the Lucas' claim that business cycles are not that costly might be true.

(b)

$$
\begin{aligned}
E_0\sum_{t=0}^{\infty}\beta^t\frac{c_t^{1-\alpha}-1}{1-\alpha} &= E_0\sum_{t=0}^{\infty}\beta^t\frac{\left[(1+\Omega)\bar{c}_0\exp\left(\sum_{s=0}^{t}z_s-\frac{t}{2}\sigma_z^2\right)\right]^{1-\alpha}-1}{1-\alpha} \\
&= [(1+\Omega)\bar{c}_0]^{1-\alpha}\sum_{t=0}^{\infty}\beta^t\frac{\exp\left[(1-\alpha)\left(\sum_{s=0}^{t}z_s-\frac{t}{2}\sigma_z^2\right)\right]}{1-\alpha}-\frac{1}{(1-\alpha)(1-\beta)}
\end{aligned}
$$

Because z_t is independently distributed across time we have that $\exp\left[(1-\alpha)\sum_{s=0}^{t}z_s\right]$

is lognormally distributed with mean $\exp[(1-\alpha)^2 \frac{t}{2}\sigma_z^2]$. Therefore,

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\alpha}-1}{1-\alpha} = [(1+\Omega)\bar{c}_0]^{1-\alpha} \sum_{t=0}^{\infty} \beta^t \frac{\exp[(1-\alpha)((1-\alpha)\frac{t}{2}\sigma_z^2 - \frac{t}{2}\sigma_z^2)]}{1-\alpha} - \frac{1}{(1-\alpha)(1-\beta)}$$

$$= \frac{[(1+\Omega)\bar{c}_0]^{1-\alpha}}{1-\alpha} \sum_{t=0}^{\infty} \left(\beta e^{-\alpha(1-\alpha)\frac{\sigma_z^2}{2}}\right)^t - \frac{1}{(1-\alpha)(1-\beta)}$$

$$= \frac{[(1+\Omega)\bar{c}_0]^{1-\alpha}}{(1-\alpha)\left(1-\beta e^{-\alpha(1-\alpha)\frac{\sigma_z^2}{2}}\right)} 1 - \frac{1}{(1-\alpha)(1-\beta)}$$

Again, the consumer is indifferent if

$$\sum_{t=0}^{\infty} \beta^t \frac{\bar{c}_t^{1-\alpha}-1}{1-\alpha} = E_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\alpha}-1}{1-\alpha}$$

$$\sum_{t=0}^{\infty} \beta^t \frac{\bar{c}_0^{1-\alpha}-1}{1-\alpha} = \frac{[(1+\Omega)\bar{c}_0]^{1-\alpha}}{(1-\alpha)\left(1-\beta e^{-\alpha(1-\alpha)\frac{\sigma_z^2}{2}}\right)} - \frac{1}{(1-\alpha)(1-\beta)}$$

$$\frac{\bar{c}_0^{1-\alpha}}{(1-\alpha)(1-\beta)} = \frac{[(1+\Omega)\bar{c}_0]^{1-\alpha}}{(1-\alpha)\left(1-\beta e^{-\alpha(1-\alpha)\frac{\sigma_z^2}{2}}\right)}$$

$$\frac{1}{(1-\beta)} = (1+\Omega)^{1-\alpha} \frac{1}{1-\beta e^{-\alpha(1-\alpha)\frac{\sigma_z^2}{2}}}$$

$$(1+\Omega)^{1-\alpha} = \frac{1-\beta e^{-\alpha(1-\alpha)\frac{\sigma_z^2}{2}}}{1-\beta}$$

$$\Omega = \exp\left(\frac{\alpha\sigma_z^2}{2}\right)\left(\frac{1-\beta e^{-\alpha(1-\alpha)\frac{\sigma_z^2}{2}}}{e^{\alpha(1-\alpha)\frac{\sigma_z^2}{2}}(1-\beta)}\right)^{\frac{1}{1-\alpha}} - 1$$

This result is different from (a) because not only the current shock but also the past shocks affect the consumption stream. In another words, shocks are persistent. If the term in the second brackets is larger than 1 (which for me is not that clear) then the welfare cost of business cycles is bigger now.

FALL 2001 - QUESTION 5

(a) To begin with, note that apart from the technological progress both sectors use the same production function (same f). Without loss of generality, assume that the production function f is Cobb-Douglas, with capital share θ and labor share $1-\theta$. The variables that must be detrended are: k, k_m, k_h, c_m, c_h. First, note that aggregate capital is produced by the market technology and then allocated to each sector. Therefore, along the BGP the stock of capital in both sectors must be growing at the same rate. Additionally, consumption in the market sector must grow at the same rate as the aggregate stock of capital. Call this rate g_m.

Use to resource constraint for the market sector and the properties of constant returns to scale technology to pin down this rate:

$$\begin{aligned} c_{mt} + k_{t+1} &= z_{mt} f(k_{mt}, X_{mt} l_{mt}) + (1-\delta)k_t \\ c_{mt} + k_{t+1} &= z_{mt}(k_{mt})^{\theta}(X_{mt} l_{mt})^{1-\theta} + (1-\delta)k_t \\ g_m^t \hat{c}_{mt} + g_m^{t+1}\hat{k}_{t+1} &= z_{mt}\left(g_m^t \hat{k}_{mt}\right)^{\theta}(e^{t\gamma_m} l_{mt})^{1-\theta} + (1-\delta)g_m^t \hat{k}_t \\ \hat{c}_{mt} + g_m \hat{k}_{t+1} &= z_{mt}(g_m^{\theta-1} e^{(1-\theta)\gamma_m})^t \hat{k}_{mt}^{\theta} l_{mt}^{1-\theta} + (1-\delta)\hat{k}_t \end{aligned}$$

This equation is stationary if

$$g_m = e^{\gamma_m} \tag{1}$$

In this case we have:

$$\hat{c}_{mt} + e^{\gamma_m}\hat{k}_{t+1} = z_{mt} f\left(\hat{k}_{mt}, l_{mt}\right) + (1-\delta)\hat{k}_t \tag{2}$$

Now lets pin down the growth rate of c_h. Call this rate g_h. Using the resource constraint for the home sector we get:

$$\begin{aligned} c_{ht} &= z_{ht} f(k_{ht}, X_{ht} l_{ht}) \\ c_{ht} &= z_{ht}\left(X_{mt}\hat{k}_{ht}\right)^{\theta}(X_{ht} l_{ht})^{1-\theta} \end{aligned}$$

This expression implies that the technological progress in the home sector is both "capital-augmenting" and "labor-augmenting". Because the production function is constant returns to scale, we can combine the factors X_m and X_h into one single factor $X = \exp[\theta\gamma_M + (1-\theta)\gamma_h]$, which is sometimes referred to as "TFP-augmenting" technological progress:

$$\begin{aligned} g_h^t \hat{c}_{ht} &= z_{ht} X_t \left(\hat{k}_{ht}^{\theta} l_{ht}^{1-\theta}\right) \\ \hat{c}_{ht} &= z_{ht}\left(\frac{e^{\theta\gamma_M + (1-\theta)\gamma_h}}{g_h}\right)^t \left(\hat{k}_{ht}^{\theta} l_{ht}^{1-\theta}\right) \end{aligned}$$

This equation is stationary if

$$g_h = e^{\theta\gamma_M + (1-\theta)\gamma_h} \tag{3}$$

Note that the growth rate of c_h is a weighted average of the technological progress in each sector. The intution is that the home sector is benefited by its own technogical progress and by the technological progress in the market sector embodied in the capital stock.

The stationary resource constraint in the home sector is:

$$\hat{c}_{ht} = z_{ht} f(\hat{k}_{ht}, l_{ht}) \quad (4)$$

Finally, lets adjuste the lifetime utility:

$$\begin{aligned}
U &\equiv E_0 \sum_{t=0}^{\infty} \beta^t [\alpha \log c_{mt} + (1-\alpha) \log c_{ht} + B \log(1 - l_{mt} - l_{ht})] \\
&= E_0 \sum_{t=0}^{\infty} \beta^t [\alpha \log(g_m^t \hat{c}_{mt}) + (1-\alpha) \log(g_h^t \hat{c}_{ht}) + B \log(1 - l_{mt} - l_{ht})] \\
&= E_0 \sum_{t=0}^{\infty} \beta^t [\alpha \log(\hat{c}_{mt}) + (1-\alpha) \log(\hat{c}_{ht}) + B \log(1 - l_{mt} - l_{ht})] + \alpha \sum_{t=0}^{\infty} \beta^t t \log g_m + (1-\alpha) \sum_{t=0}^{\infty} \beta^t t \log g_h \\
&= E_0 \sum_{t=0}^{\infty} \beta^t [\alpha \log(\hat{c}_{mt}) + (1-\alpha) \log(\hat{c}_{ht}) + B \log(1 - l_{mt} - l_{ht})] + \frac{\beta}{(1-\beta)^2} (\alpha \log g_m + (1-\alpha) \log g_h)
\end{aligned}$$

From now on we can ignore the constant term.

Using the previous equations, we can write down the planner's DPP as follows:

$$v(k, z_m, z_h) = \max_{\hat{k}', \hat{k}_m, l_m, l_h} \{\alpha \log(\hat{c}_m) + (1-\alpha) \log(\hat{c}_h) + B \log(1 - l_m - l_h) + \beta E[v(k', z_m', z_h')]\}$$

$$\begin{aligned}
\text{s.t. } \hat{c}_m &= z_m f\left(\hat{k}_m, l_m\right) + (1-\delta)\hat{k} - e^{\gamma_m} \hat{k}' \\
\hat{c}_h &= z_h f\left(\hat{k} - \hat{k}_m, l_h\right) \\
z_m' &= (1-\rho) + \rho z_{m-1} + \varepsilon_m \\
z_h' &= (1-\rho) + \rho z_{h-1} + \varepsilon_h
\end{aligned}$$

(b) I guess there is a mistake in the question. Where you read "are also necessary conditions" we should read "are also sufficient conditions". Since the objective function is strictly concave we require the opportunity sets to be convex for the FOCs to be also sufficient conditions for a maximum. The shape of the opportunity sets depend on the assumption you make for the production functions. One sufficient assumption would be that both production functions are concave in each argument, for example the Cobb-Douglas case.

The planner's FOCs are:

$$
\begin{aligned}
l_m &: \frac{B}{1-l_m-l_h} = \frac{\alpha z_m f_2\left(\hat{k}_m, l_m\right)}{\hat{c}_m} \\
l_h &: \frac{B}{1-l_m-l_h} = \frac{(1-\alpha) z_h f_2\left(\hat{k}-\hat{k}_m, l_h\right)}{\hat{c}_h} \\
k_m &: \frac{\alpha z_m f_1\left(\hat{k}_m, l_m\right)}{\hat{c}_m} = \frac{(1-\alpha) z_h f_1\left(\hat{k}-\hat{k}_m, l_h\right)}{\hat{c}_h} \\
k' &: \frac{\alpha e^{\gamma_m}}{\hat{c}_m} = \beta E\left[\frac{\alpha(1-\delta)}{\hat{c}_m'} + \frac{(1-\alpha) z_h' f_1\left(\hat{k}'-\hat{k}_m', l_h'\right)}{\hat{c}_h'}\right]
\end{aligned}
$$

(c) Household's DPP:

$$
v(\hat{K},\hat{k},z_m,z_h) = \max_{\hat{k}',l,\hat{c}_h}\left\{\alpha\log(\hat{c}_m) + (1-\alpha)\log(\hat{c}_h) + B\log(1-l) + \beta E[v(\hat{K}',\hat{k}',z_m',z_h')\right\} \tag{5}
$$

$$
\begin{aligned}
\text{s.t. } \hat{c}_m &= w(\hat{K},z_m,z_h)l + r(\hat{K},z_m,z_h)\hat{k} + (1-\delta)\hat{k} - e^{\gamma_m}\hat{k}' - p(\hat{K},z_m,z_h)\hat{c}_h \\
\hat{K}' &= G(\hat{K},z_m,z_h) \\
z_m' &= (1-\rho) + \rho z_{m-1} + \varepsilon_m \\
z_h' &= (1-\rho) + \rho z_{h-1} + \varepsilon_h
\end{aligned}
$$

where p is the relative price of the home consumption good. Note: I am assuming that there is perfect mobility of factors across sectors so that the wage rate and the rental rate of capital are the same in both sectors.

Home consumption goods producer:

$$
\max_{\hat{K}_h,L_h}\left\{p(\hat{K},z_m,z_h)z_h f\left(\hat{K}_h, L_h\right) - w(\hat{K},z_m,z_h)L_h - r(\hat{K},z_m,z_h)\hat{K}_h\right\} \tag{6}
$$

Market goods producer:

$$
\max_{\hat{K}_m,L_m}\left\{z_m f\left(\hat{K}_m, L_m\right) - w(\hat{K},z_m,z_h)L_m - r(\hat{K},z_m,z_h)\hat{K}_m\right\} \tag{7}
$$

Note: I am using capital letters to denote firms' choice variables.

Definition: A recursive competitive equilibrium for this economy is:

(i) Decision rules $\hat{k}'(\hat{K},\hat{k},z_m,z_h), l(\hat{K},\hat{k},z_m,z_h), \hat{c}_h(\hat{K},\hat{k},z_m,z_h)$ for the household

(ii) Decision rules $\hat{K}_h(\hat{K},z_m,z_h), L_h(\hat{K},z_m,z_h)$ for the home consumption goods producer

(iii) Decision rules $\hat{K}_m(\hat{K},z_m,z_h), L_m(\hat{K},z_m,z_h)$ for the market goods producer

(iv) Pricing functions $w(\hat{K},z_m,z_h), r(\hat{K},z_m,z_h), p(\hat{K},z_m,z_h)$

(v) An aggregate law of motion $\hat{K}' = G(\hat{K},z_m,z_h)$

such that

(1) Given (iv) and (v), (i) solves problem (5)

(2) Given (iv), (ii) solves problem (6)

(3) Given (iv), (iii) solves problem (7)

(4) Markets clear:

$$\begin{aligned} l(\hat{K},\hat{K},z_m,z_h) &= L_m(\hat{K},z_m,z_h) + L_h(\hat{K},z_m,z_h) \\ \hat{K} &= \hat{K}_m(\hat{K},z_m,z_h) + \hat{K}_h(\hat{K},z_m,z_h) \\ \hat{c}_h(\hat{K},\hat{K},z_m,z_h) &= z_h f\big(\hat{K}_h(\hat{K},z_m,z_h), L_h(\hat{K},z_m,z_h)\big) \end{aligned}$$

By Walras' law the market goods also clears.

(5) Perceptions are correct:

$$\hat{k}'(\hat{K},\hat{K},z_m,z_h) = G(\hat{K},z_m,z_h)$$

(d) Overall, the economy is more volatile with $\alpha = 1/2$ than with $\alpha = 1$. In the latter case the home sector is active and the aggregate economy is hit by two different shocks. Moreover, the household has more margins for substitution. For example, if there is a positive shock in the home sector and a negative shock in the market sector, then resources will be reallocated to the sector with the positive shock. If the economy has only one active sector (case of $\alpha = 1$) this cross sector substitution becomes impossible. I leave the details of what happens with each variable to you.

SPRING 2001 - QUESTION 1

(a) Planner's DPP:

$$v(K,\Gamma) = \max_{K',H}\left\{\frac{[C^{\alpha}(1-H)^{1-\alpha}]^{1-\sigma}-1}{1-\sigma} + \beta\eta v(K',\Gamma')\right\}$$

$$\text{s.t. } C + \eta K' = \Gamma^{1-\theta}K^{\theta}H^{1-\theta} + (1-\delta)K$$

$$\Gamma' = \gamma\Gamma$$

This problem is well defined if $\beta\eta < 1$.

(b) From the resource constraint you can figure out that along the BGP all variables grow at the same and constant rate $g = \gamma$. I leave to you the full characterization of the BGP.

(c) First, find the equations that characterize the BGP steady state of the economy:

$$\text{Production Function}: \hat{Y} = \hat{K}^{\theta} H^{1-\theta}$$
$$\text{Resource Constraint}: \hat{C} + \hat{I} = \hat{Y}$$
$$\text{Law of Motion of Capital}: \hat{I} = (\eta\gamma + \delta - 1)\hat{K}$$
$$\text{Labor Choice}: (1-\alpha)\frac{H}{1-H} = \alpha(1-\theta)\frac{\hat{Y}}{\hat{C}}$$
$$\text{Euler Equation}: \gamma = \beta\left(\theta\frac{\hat{Y}}{\hat{K}} + 1 - \delta\right)$$

Note: I am using hat to denote stationary variables.

The **quarterly** data we have are:

$$\text{(i)}: g_y = (1.014)^{1/4} = 1.0035$$
$$\text{(ii)}: g_n = (1.015)^{1/4} = 1.0037$$
$$\text{(iii)}: \frac{r\hat{K}}{\hat{Y}} = 0.4$$
$$\text{(iv)}: \frac{\hat{I}}{\hat{Y}} = .25$$
$$\text{(v)}: \frac{\hat{K}}{\hat{Y}} = 4 * 3.5 = 14$$
$$\text{(v)}: H = .31$$
$$\text{(vi)}: \sigma = 1.5$$

The parameters we need to calibrate are: $\alpha, \beta, \gamma, \delta, \eta, \theta$. Note that γ, η and θ are calibrated directly from facts (i), (ii) and (iii), respectively:

$$\gamma = 1.0035$$
$$\eta = 1.0037$$
$$\theta = 0.4$$

To calibrate δ use the law of motion of capital and facts (iv) and (v):

$$\delta = \frac{\hat{I}}{\hat{K}} + 1 - \eta\gamma = \frac{\hat{I}/\hat{Y}}{\hat{K}/\hat{Y}} + 1 - \eta\gamma = \frac{.25}{14} + 1 - (1.0035)(1.0037) = 0.010644$$

Use the values of γ, δ and θ, fact (v) and the euler equation to calibrate β :

$$\beta = \frac{\gamma}{\theta\frac{\hat{Y}}{\hat{K}} + 1 - \delta} = \frac{1.0035}{.4\frac{1}{14} + 1 - 0.010644} = 0.98583$$

Finally, to calibrate α use (v) and the labor choice equation:

$$\alpha = \frac{H/(1-H)}{(1-\theta)\frac{\hat{Y}}{\hat{C}} + H/(1-H)} = \frac{H}{(1-\theta)(1-H)\frac{1}{1-\hat{I}/\hat{Y}} + H} = \frac{.31}{0.6 * .69 * \frac{1}{.75} + .31} = 0.35963$$

SPRING 2001 - QUESTION 2

(a) The problem solved by country i is:

$$\max \sum_{t=0}^{\infty} \beta_i^t \log(k_{it}^{\alpha} + (1-\delta)k_{it} - k_{it+1}), \quad i = 1,2$$

The capital euler equation for country i is:

$$\frac{c_{it+1}}{c_{it}} = \beta_i(\alpha k_{it+1}^{\alpha-1} + 1 - \delta), \quad i = 1,2$$

In steady state we have:

$$\frac{1}{\beta_i} = MPk_i + 1 - \delta = R_i, \quad i = 1,2$$

where R_i is the return on capital and MPk_i is the marginal product of capital in country i.

Solving for the per capita capital $\bar{k}_i$ and per capita income $\bar{y}_i$:

$$\bar{k}_i = \left(\frac{\alpha}{1/\beta_i - (1-\delta)}\right)^{\frac{1}{1-\alpha}}, \quad i = 1,2$$
$$\bar{y}_i = \left(\frac{\alpha}{1\beta_i - (1-\delta)}\right)^{\frac{\alpha}{1-\alpha}}, \quad i = 1,2$$

If there is no capital mobility, each produces and consumes in autarky. Using the parameter values we get:

$$\bar{k}_1 = 6.0858$$
$$\bar{k}_2 = 2.5201$$
$$\bar{y}_1 = 1.8257$$
$$\bar{y}_2 = 1.3608$$

Note that although the technology is the same, country 1 managed to accumulate more capital and get a higher per capita income because it is more patient (higher discount factor). Also note that the steady state marginal product of capital and hence the return on capital is higher in country 2:

$$1.02 = R_1 = MPk_1 + 1 - \delta < MPk_2 + 1 - \delta = R_2 = 1.10$$

(b) With perfect capital mobility, the low-return country 1 can export some of its capital to the high-return country 2. The capital outflow will raise the marginal product of capital in country 1, and the capital inflow will reduce the marginal product of capital in country 2. This process continues until the return of capital is the same in both countries, that is until

$$R_1 = R_2 = R$$

where R is the common rate of return in steady state.

The problem solved for country i is now given by:

$$\max \sum_{t=0}^{\infty} \beta_i^t \log(k_{it}^{\alpha} + (1-\delta)k_{it} - k_{it+1} + b_{it+1} - R_t b_{it}), \quad i = 1,2$$

where b_i denotes imports of capital (if positive) or exports of capital (if negative). Obviously,

$$b_1 + b_2 = 0 \Rightarrow b_2 = -b_1$$

Now we have 2 eulers equation for country i:

$$\frac{c_{it+1}}{c_{it}} = \beta_i(\alpha k_{it+1}^{\alpha-1} + 1 - \delta), \quad i = 1,2$$
$$\frac{c_{it+1}}{c_{it}} = \beta_i R_t, \quad i = 1,2$$

In steady state we have

$$1 = \beta_i(\alpha \bar{k}_i^{\alpha-1} + 1 - \delta), \quad i = 1,2 \tag{1}$$
$$1 = \beta_i R, \quad i = 1,2 \tag{2}$$

From the above argument we know that

$$\alpha \bar{k}_1^{\alpha-1} + 1 - \delta = \alpha \bar{k}_2^{\alpha-1} + 1 - \delta = R$$

Hence, equations (1) and (2) collapse to just one condition:

$$\frac{1}{\beta_i} = R, \quad i = 1,2 \tag{3}$$

The problem is that equation (3) cannot hold simultaneously because the discount factors are different. This leaves us with three possible situations:

$$(i) : \frac{1}{\beta_1} < R < \frac{1}{\beta_2}$$
$$(ii) : \frac{1}{\beta_1} < R = \frac{1}{\beta_2}$$
$$(iii) : \frac{1}{\beta_1} = R < \frac{1}{\beta_2}$$

Note that these three cases compare the marginal rate of substitution (or marginal cost) of each country $1/\beta$ to the return on capital (or marginal benefit) R. We can dismiss (i) and (ii) because they imply that country 2 will accumulate too much capital. Besides violating the TVC, this would drive the marginal product of capital in this country to zero. Hence, country 2 would be better off by exporting some capital to country 2. Therefore, (iii) must be the equilibrium. The intuition is that country 1 will accumulate enough capital to afford its own consumption and investment and still borrow some of its capital to country 2. Since country 2 is less patient it will not accumulate any capital (zero investment). Instead, it will borrow from country 2 all the capital it needs and will use this capital to produce its own consumption and to pay back the capital borrowed plus the return R. In the limit, country 1 will own the entire capital of the world and country 2 will own nothing. However, because of trade both countries can consume more and are better off than in autarky.

Since the euler equation holds for country 1 we have:

$$\frac{1}{\beta_1} = \alpha \bar{k}_1^{\alpha-1} + 1 - \delta \Rightarrow \bar{k}_1 = \left(\frac{\alpha}{1/\beta_1 - (1-\delta)} \right)^{\frac{1}{1-\alpha}} = 6.0858$$

This implies that the marginal product of capital is

$$r = 0.10$$

Because country 2 does not accumulate capital we have:

$$\bar{k}_2 = 0$$

The capital account is given by:

$$\begin{aligned} \bar{k}_1 + \bar{b}_1 &= \bar{k}_2 + \bar{b}_2 = \bar{k}_2 - \bar{b}_1 \\ &\Rightarrow \bar{b}_1 = -\frac{\bar{k}_1}{2} \\ &\Rightarrow \bar{b}_2 = \frac{\bar{k}_1}{2} \end{aligned}$$

Also,

$$\begin{aligned} \bar{y}_1 &= GDP_1 = \bar{k}_1^{\alpha} = 1.8257 \\ GNP_1 &= GDP_1 - r\bar{b}_1 = 1.8257 - 0.10(-3.0429) = 2.1300 \\ \bar{y}_2 &= GDP_2 = \left(\frac{\bar{k}_1}{2} \right)^{\alpha} = 1.4491 \\ GNP_2 &= GDP_2 - r\bar{b}_2 = 1.4491 - 0.10(3.0429) = 1.1448 \end{aligned}$$

(c) Capital mobility reduces per capita GNP for country 2 from 1.3608 to 1.1448. (note that $GNP = GDP$ in autarky), although it improves per capita GDP, and its share of world's output from $1.3608/(1.3608 + 1.8257) = .427$ to $1.4491/(1.$

$449\,1 + 1.8257) = .4425.$

SPRING 2001 - QUESTION 5

(a) To begin with, note that the transition matrix assumed here implies that the shocks are not only independently but also identically distributed. The joint distribution of shocks (conditional and unconditional) is:

$$\Pr\left(\gamma = \gamma^i, z_m = z_m^j, z_h = z_h^k\right) = (0.5)^3 = 0.125$$

where
$(i,j,k) \in \{(H,H,H),(H,H,L),(H,L,H),(H,L,L),(L,H,H),(L,H,L),(L,L,H),(L,L,L)\}$, $H =$ high and $L =$ low.

Instead of writing down the planner's DPP, in which we have to deal with 8 value functions, it is easier to set up the sequential problem:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left[(\gamma_t z_{mt} l_{mt})^{\alpha} (\gamma_t z_{ht} l_{ht})^{1-\alpha} - B(l_{mt} + l_{ht}) \right]$$

$$= \max \sum_{t=0}^{\infty} \beta^t \sum_{(i,j,k)} \Pr\left(\gamma = \gamma^i, z_m = z_m^j, z_h = z_h^k\right)\left[\gamma^i \left(z_m^j l_m^j\right)^{\alpha} (z_h^k l_h^k)^{1-\alpha} - B\left(l_m^j + l_h^k\right)\right]$$

Given $\left(\gamma = \gamma^i, z_m = z_m^j, z_h = z_h^k\right)$, the FOCs are:

$$l_m^j : B = \frac{\alpha\gamma^i \left(z_m^j l_m^j\right)^{\alpha} (z_h^k l_h^k)^{1-\alpha}}{l_m^j}$$
$$l_h^k : B = \frac{(1-\alpha)\gamma^i \left(z_m^j l_m^j\right)^{\alpha} (z_h^k l_h^k)^{1-\alpha}}{l_h^k}$$

These imply:

$$\frac{\alpha}{1-\alpha} = \frac{l_m^j}{l_h^k}$$

Combine this equation with the time constraint $l_m^j + l_h^k = 1$ to get:

$$l_m^j = \alpha$$
$$l_h^k = 1 - \alpha$$

(b), (c) The time spent working in each sector is constant, regardless the aggregate and sectoral shocks. However, sectoral consumption responds to both shocks.

FALL 2002 - QUESTION 1

(a) Suppose there is a benevolent social planner that maximizes the welfare of the representative agent. Let n be the fraction of employed agents. The planner will maximize the following weighted average utility function:

$$
\begin{aligned}
U(c_1, c_2, n) &= n\frac{\left(c_1^{\alpha}(1-\bar{h})^{1-\alpha}\right)^{1-\sigma}-1}{1-\sigma} + (1-n)\frac{\left(c_2^{\alpha}(1-0)^{1-\alpha}\right)^{1-\sigma}-1}{1-\sigma} \\
&= n\frac{\left(c_1^{\alpha}(1-\bar{h})^{1-\alpha}\right)^{1-\sigma}-1}{1-\sigma} + (1-n)\frac{c_2^{\alpha(1-\sigma)}-1}{1-\sigma}
\end{aligned}
$$

where c_1 is the consumption of employed agents, and c_2 is the consumption of unemployed agents. Note that **total hours worked** are $H \equiv n\bar{h}$.

The aggregate resource constraint is:

$$
nc_1 + (1-n_t)c_2 + K' = e^z K^{\theta}\left(n\bar{h}\right)^{1-\theta} + (1-\delta)K \tag{1}
$$

Bellman equation:

$$
\begin{aligned}
v(K,z) &= \max_{K',n,c_1,c_2}\left\{ n\frac{\left(c_1^{\alpha}(1-\bar{h})^{1-\alpha}\right)^{1-\sigma}-1}{1-\sigma} + (1-n)\frac{c_2^{\alpha(1-\sigma)}-1}{1-\sigma} + \beta E[v(K',z')]\right\} \\
nc_1 + (1-n)c_2 + K' &= e^z K^{\theta}\left(n\bar{h}\right)^{1-\theta} + (1-\delta)K \\
z' &= \rho_z z + \varepsilon'
\end{aligned}
$$

(b) Combining the FOCs with respect to c_1 and c_2 gives:

$$
\frac{c_2}{c_1} = (1-\bar{h})^{\frac{(1-\alpha)(1-\sigma)}{\alpha(1-\sigma)-1}} \tag{2}
$$

The FOC with respect to n is:

$$
\left(1-\frac{c_2}{c_1}\right)\left(\alpha - \frac{1}{1-\sigma}\right) = \frac{\alpha(1-\theta)e^z K^{\theta}\left(n\bar{h}\right)^{1-\theta}}{nc_1} \tag{3}
$$

Combining the FOC with respect to K' and the EC yields the following capital euler equation:

$$
1 = \beta E\left[\left(\frac{c_1'}{c_1}\right)^{\alpha(1-\sigma)-1}\left(\theta e^{z'}K'^{\theta-1}\left(n'\bar{h}\right)^{1-\theta} + 1 - \delta\right)\right] \tag{4}
$$

The solution to the planner's problem is characterized by equations (1)-(4).

(c) The answer depends of the size of the coefficient of risk aversion σ. Using equation (2) we have the following relevant cases:

$$
\begin{aligned}
\sigma = 1 &\Rightarrow c_1 = c_2 \\
\sigma > 1 &\Rightarrow c_1 > c_2
\end{aligned}
$$

(d) Without loss of generality assume $\sigma = 1$ (You can try the other case as well). This implies the following utility function (see Hansen's HW 4):

$$
\begin{aligned}
U &= \alpha \log c_t + n_t(1-\alpha)\log(1-\bar{h}) \\
&= \alpha \log c_t - A(n_t\bar{h}) && \text{where } A \equiv -\frac{(1-\alpha)\log(1-\bar{h})}{\bar{h}} \\
&= \alpha \log c_t - AH_t && \text{where } H_t \equiv n_t\bar{h}
\end{aligned}
$$

Now, the dynamic system of equations is given by:

$$
\begin{aligned}
\text{Production function} &: y_t = e^{z_t}K_t^{\theta}H_t{}^{1-\theta} \\
\text{Resource constraint} &: c_t + K_{t+1} = y_t + (1-\delta)K_t \\
\text{Labor euler equation} &: A = \alpha(1-\theta)\frac{y_t}{c_tH_t} \\
\text{Capital euler equation} &: 1 = \beta E_t\Big[\Big(\frac{c_t}{c_{t+1}}\Big)\Big(\theta\frac{y_{t+1}}{K_{t+1}} + 1 - \delta\Big)\Big]
\end{aligned}
$$

This system of equations determine c_t, H_t, K_t and y_t. Let bar variables denote steady states, and let hat variables denote log deviations from the steady state. Taking this into account and using the log linearization rules, we obtain the following log-linearized system:

$$0 = \hat{y}_t - z_t - \theta\hat{K}_t - (1-\theta)\hat{H}_t \tag{5.1}$$

$$0 = \hat{y}_t + (1-\delta)\frac{\bar{K}}{\bar{y}}\hat{K}_t - \frac{\bar{c}}{\bar{y}}c_t - \frac{\bar{K}}{\bar{y}}\hat{K}_{t+1} \tag{5.2}$$

$$0 = \hat{y}_t - \hat{c}_t - \hat{H}_t \tag{5.3}$$

$$0 = E_t\Big(\hat{c}_t - \hat{c}_{t+1} + \beta\theta\frac{\bar{y}}{\bar{K}}(\hat{y}_{t+1} - \hat{K}_{t+1})\Big) \tag{5.4}$$

(e) Note that the first three equations can be used to solve for $\hat{y}_t, \hat{H}_t$ and $\hat{c}_t$ in terms of $\hat{K}_t, \hat{K}_{t+1}$ and z_t. Using matrix notation:

$$
\begin{pmatrix} 1 & 0 & -(1-\theta) \\ 1 & -\frac{\bar{c}}{\bar{y}} & 0 \\ 1 & -1 & -1 \end{pmatrix}
\begin{pmatrix} \hat{y}_t \\ \hat{c}_t \\ \hat{H}_t \end{pmatrix}
=
\begin{pmatrix} z_t + \theta\hat{K}_t \\ \frac{\bar{K}}{\bar{y}}\hat{K}_{t+1} - (1-\delta)\frac{\bar{K}}{\bar{y}}\hat{K}_t \\ 0 \end{pmatrix}
$$

This subsystem of equations has a solution as long as the matrix

$$
\begin{pmatrix} 1 & 0 & -(1-\theta) \\ 1 & -\frac{\bar{c}}{\bar{y}} & 0 \\ 1 & -1 & -1 \end{pmatrix}
$$

is invertible. This matrix is the CC matrix in Uhlig's program. That's why Uhlig's program requires the CC matrix be full rank and therefore invertible. After you solve this subsystem, you plug the solution into the euler equation. This procedure simplifies a lot the solution because we end up with just one stochastic difference equation that we can handle using the method of undetermined coefficients (MUC).

To illustrate once more how MUC works, I will solve the problem step by step I guess you don't have to do this in the exam). First, lets compute the solution for the subsystem (5.1)-(5.3). If you work on the algebra a little bit you will get (warning: always check the algebra because I usually get it wrong):

$$\hat{H}_t = \frac{1}{1-\theta\frac{\bar{\imath}}{\bar{y}}}\left[\frac{\bar{K}}{\bar{y}}\hat{K}_{t+1} - \frac{\bar{\imath}}{\bar{y}}z_t - \left(\theta\frac{\bar{\imath}}{\bar{y}} + (1-\delta)\frac{\bar{K}}{\bar{y}}\right)\hat{K}_t\right]$$

$$\hat{c}_t = \frac{1}{1-\theta\frac{\bar{\imath}}{\bar{y}}}\left[-\theta\frac{\bar{K}}{\bar{y}}\hat{K}_{t+1} + z_t + \theta\left(1 + (1-\delta)\frac{\bar{K}}{\bar{y}}\right)\hat{K}_t\right]$$

$$\hat{y}_t = \frac{1}{1-\theta\frac{\bar{\imath}}{\bar{y}}}\left[(1-\theta)\frac{\bar{K}}{\bar{y}}\hat{K}_{t+1} + \frac{\bar{c}}{\bar{y}}z_t + \left(\theta\frac{\bar{c}}{\bar{y}} - (1-\theta)(1-\delta)\frac{\bar{K}}{\bar{y}}\right)\hat{K}_t\right]$$

where $\bar{c}/\bar{y}$ is the steady state consumption-output ratio, $\bar{\imath}/\bar{y}$ is the steady state investment-output ratio, and $\bar{K}/\bar{y}$ is the steady state capital-output ratio.

To economize on notation, rewrite the above system as follows:

$$\hat{H}_t = h_1 z_t + h_2\hat{K}_t + h_3\hat{K}_{t+1} \tag{6.1}$$

$$\hat{c}_t = c_1 z_t + c_2\hat{K}_t + c_3\hat{K}_{t+1} \tag{6.2}$$

$$\hat{y}_t = y_1 z_t + y_2\hat{K}_t + y_3\hat{K}_{t+1} \tag{6.3}$$

where

$$h_1 \equiv \frac{-\bar{\imath}/\bar{y}}{1-\theta\frac{\bar{\imath}}{\bar{y}}}; h_2 \equiv \frac{-1}{1-\theta\frac{\bar{\imath}}{\bar{y}}}\left(\theta\frac{\bar{\imath}}{\bar{y}} + (1-\delta)\frac{\bar{K}}{\bar{y}}\right); h_3 \equiv \frac{\bar{K}/\bar{y}}{1-\theta\frac{\bar{\imath}}{\bar{y}}}$$

$$c_1 \equiv \frac{1}{1-\theta\frac{\bar{\imath}}{\bar{y}}}; c_2 \equiv \frac{\theta}{1-\theta\frac{\bar{\imath}}{\bar{y}}}\left(1 + (1-\delta)\frac{\bar{K}}{\bar{y}}\right); c_3 \equiv \frac{-\theta\bar{K}/\bar{y}}{1-\theta\frac{\bar{\imath}}{\bar{y}}}$$

$$y_1 \equiv \frac{\bar{c}/\bar{y}}{1-\theta\frac{\bar{\imath}}{\bar{y}}}; y_2 \equiv \frac{-1}{1-\theta\frac{\bar{\imath}}{\bar{y}}}\left(\theta\frac{\bar{c}}{\bar{y}} - (1-\theta)(1-\delta)\frac{\bar{K}}{\bar{y}}\right); y_3 \equiv \frac{(1-\theta)\bar{K}/\bar{y}}{1-\theta\frac{\bar{\imath}}{\bar{y}}}$$

Substitute (6.1)-(6.3) into the capital euler equation:

$$\begin{aligned} 0 &= E_t\left[c_1(z_t - z_{t+1}) + c_2\left(\hat{K}_t - \hat{K}_{t+1}\right) + c_3\left(\hat{K}_{t+1} - \hat{K}_{t+2}\right) + \beta\theta\frac{\bar{y}}{\bar{K}}(y_1 z_{t+1} + y_2\hat{K}_{t+1} + y_3\hat{K}_{t+2} - \hat{K}_{t+1})\right] \\ &= E_t\left[\alpha_1 z_t + \alpha_2 z_{t+1} + \alpha_3\hat{K}_t + \alpha_4\hat{K}_{t+1} + \alpha_5\hat{K}_{t+2}\right] \end{aligned} \tag{7}$$

where

$$
\begin{aligned}
\alpha_1 &= c_1 \\
\alpha_2 &= \beta\theta\frac{\bar{y}}{\bar{K}}y_1 - c_1 \\
\alpha_3 &= c_2 \\
\alpha_4 &= \beta\theta\frac{\bar{y}}{\bar{K}}(y_2 - 1) + c_3 - c_2 \\
\alpha_5 &= \beta\theta\frac{\bar{y}}{\bar{K}}y_3 - c_3
\end{aligned}
$$

Conjecture that the law of motion of capital is of the form:

$$\hat{K}_{t+1} = \gamma_1 z_t + \gamma_2 \hat{K}_t \tag{8}$$

Substitute this conjecture and the law of motion of z into (7):

$$0 = E_t\left[\alpha_1 z_t + \alpha_2(\rho z_t + \varepsilon_{t+1}) + \alpha_3\hat{K}_t + \alpha_4\left(\gamma_1 z_t + \gamma_2\hat{K}_t\right) + \alpha_5\left(\gamma_1 z_{t+1} + \gamma_2\hat{K}_{t+1}\right)\right] \qquad \#$$

$$= [\alpha_1 + \alpha_2\rho + \gamma_1(\alpha_4 + \alpha_5\rho + \gamma_2\alpha_5)]z_t + [\alpha_3 + \gamma_2(\alpha_4 + \alpha_5\gamma_2)]\hat{K}_t \tag{9}$$

This equation holds in all t if

$$
\begin{aligned}
\alpha_1 + \alpha_2\rho + \gamma_1(\alpha_4 + \alpha_5\rho + \gamma_2\alpha_5) &= 0 \\
\alpha_3 + \gamma_2(\alpha_4 + \alpha_5\gamma_2) &= 0
\end{aligned}
$$

Note that we are using the MUC here because (9) is equivalent to:

$$0z_t + 0\hat{K}_t = [\alpha_1 + \alpha_2\rho + \gamma_1(\alpha_4 + \alpha_5\rho + \gamma_2\alpha_5)]z_t + [\alpha_3 + \gamma_2(\alpha_4 + \alpha_5\gamma_2)]\hat{K}_t$$

Solving for γ_1 and γ_2 gives:

$$
\begin{aligned}
\gamma_2 &= \frac{1}{2\alpha_5}\left[-\alpha_4 \pm \sqrt{\alpha_4 - 4\alpha_3\alpha_5}\right] \\
\gamma_1 &= -\frac{\alpha_1 + \alpha_2\rho}{\alpha_4 + \alpha_5(\rho + \gamma_2)}
\end{aligned}
$$

In the case of the growth model, the solution is usually a saddle path. This means that both γ_2 are real, but one lies inside the unit circle and the other lies outside the unit circle (in absolute value). It is at this point that the TVC comes into play. The TVC requires that you pick up the γ_2 that is smaller than one in absolute value. This ensures that the law of motion of capital (8) is not explosive.

(f) The planner's problem can be decentralized if we assume that workers trade lotteries specifying the probability of work. This lottery (remember that it is equivalent to unemployment insurance) is the additional commodity. See Hansen's HW4 for the decentralization.

FALL 2002 - QUESTION 2

(a) The household solves the following dynamic programming problem:

$$v(K,k) = \max_{c,k'}\{u(c) + \beta v(K',k')\} \quad (1)$$

$$\text{s.t. } c = w(K) + r(K)k - q(K)[k' - (1-\delta)k]$$

$$K' = G(K)$$

where q is the relative price of investment goods.

The consumption goods producer solves the following problem:

$$\max_{(K_{1t}^f, H_{1t}^f)} \left\{F^1\left(K_1^f, H_1^f\right) - r(K)K_1^f - w(K)H_1^f\right\} \quad (2)$$

The investment goods producer solves the following problem:

$$\max_{(K_2^f, H_2^f)} \left\{q(K)F^2\left(K_2^f, H_2^f\right) - r(K)K_2^f - w(K)H_2^f\right\} \quad (3)$$

A recursive competitive equilibrium for this economy is:

(i) A set of policy functions $k'(K,k)$ and $c(K,k)$ for the household

(ii) A set of decision rules $K_1^f(K)$ and $H_1^f(K)$ for the consumption goods producer

(iii) A set of decision rules $K_2^f(K)$ and $H_2^f(K)$ for the investment goods producer

(iv) Pricing functions $r(K),\ w(K)$ and $q(K)$

(v) A law of motion for the aggregate state $K' = G(K)$

such that

(1) Given (iv) and (v), (i) solves the household problem (1)

(2) Given (iv), (ii) solves the problem of the consumption goods producer (2)

(3) Given (iv), (iii) solves the problem of the investment goods producer (3)

(4) Markets clear:

Labor market: $H_1^f(K) + H_2^f(K) = 1$

Capital market: $K_1^f(K) + K_2^f(K) = K$

Investment goods market: $k'(K,K) - (1-\delta)K = F^2[K_2^f(K), H_2^f(K)]$

Consumption goods market: $c(K,K) = F^1[K_1^f(K), H_1^f(K)]$

(5) Perceptions are correct:

$$k'(K,K) = G(K)$$

(b) With adjustment costs we have that capital become sector specific. The state variables are (K_1, K_2) and $(K_{1_{-1}}, K_{2_{-1}})$. Define $K \equiv (K_1, K_2)$ and $K_{-1} \equiv (K_{1_{-1}}, K_{2_{-1}})$. (Note that the adjustment cost is in units of the consumption goods).

Household's problem:

$$v(K, K_{-1}, k, k_{-1}) = \max_{c, k_1', k_2'} \{u(c) + \beta v(K', K, k', k)\} \tag{4}$$

$$\text{s.t. } c = w_1(K, K_{-1}) + w_2(K, K_{-1}) + r_1(K, K_{-1})k_1 + r_2(K, K_{-1})k_2$$
$$- q(K, K_{-1})[k_1' + k_2' - (1-\delta)(k_1 + k_2)] - h\left(\frac{k_1}{k_{1_{-1}}}\right) - \left(\frac{k_2}{k_{2_{-1}}}\right)$$
$$K_1' = G_1(K, K_{-1})$$
$$K_2' = G_2(K, K_{-1})$$

Consumption goods producer problem:

$$\max_{(K_{1t}^f, H_{1t}^f)} \left\{F^1\left(K_1^f, H_1^f\right) - r_1(K, K_{-1})K_1^f - w_1(K, K_{-1})H_1^f\right\} \tag{5}$$

Investment goods producer problem:

$$\max_{(K_2^f, H_2^f)} \left\{q(K, K_{-1})F^2\left(K_2^f, H_2^f\right) - r_2(K, K_{-1})K_2^f - w_2(K, K_{-1})H_2^f\right\} \tag{6}$$

A recursive competitive equilibrium for this economy is:

(i) A set of policy functions $k_1'(K, K_{-1}, k, k_{-1}), k_2'(K, K_{-1}, k, k_{-1})$ and $c(K, K_{-1}, k, k_{-1})$ for the household

(ii) A set of decision rules $K_1^f(K, K_{-1})$ and $H_1^f(K, K_{-1})$ for the consumption goods producer

(iii) A set of decision rules $K_2^f(K, K_{-1})$ and $H_2^f(K, K_{-1})$ for the investment goods producer

(iv) Pricing functions $r_1(K, K_{-1})$, $r_2(K, K_{-1}), w_1(K, K_{-1})$, $w_2(K, K_{-1})$ and $q(K, K_{-1})$

(v) Laws of motion for the aggregate states $K_1' = G_1(K, K_{-1})$ and $K_2' = G_2(K, K_{-1})$

such that

(1) Given (iv) and (v), (i) solves the household problem (4)

(2) Given (iv), (ii) solves the problem of the consumption goods producer (5)

(3) Given (iv), (iii) solves the problem of the investment goods producer (6)

(4) Markets clear:

Labor market: $H_1^f(K, K_{-1}) + H_2^f(K, K_{-1}) = 1$

Capital market: $K_1^f(K, K_{-1}) = K_1$ and $K_2^f(K, K_{-1}) = K_2$

Investment goods market: $K_1' + K_2' - (1-\delta)(K_1 + K_2) = F^2[K_2^f(K, K_{-1}), H_2^f(K, K_{-1})]$

Consumption goods market: $c(K,K_{-1},K,K_{-1}) + h\left(\frac{K_1}{K_{1_{-1}}}\right) + \left(\frac{K_2}{K_{2_{-1}}}\right) = F^1[K_1^f(K,K_{-1}),H_1^f(K,K_{-1})]$

(5) Perceptions are correct:

$$k_1'(K,K_{-1},K,K_{-1}) = K_1' = G_1(K,K_{-1})$$
$$k_2'(K,K_{-1},K,K_{-1}) = K_2' = G_2(K,K_{-1})$$

FALL 2002 - QUESTION 5

(a) First, it is convenient to make the model stationary. The procedure here will differ a bit from the standard one-sector growth model because we have two types of technological progress. Therefore, we cannot expect that all variables will grow at the same rate along the BGP. Lets conjecture that along the BGP consumption will grow at the rate g_c and investment, hence capital as well, will grow at the rate g_i.

Conjectures:

$$I_t = (1+g_i)^t\hat{I}_t;\quad K_t = (1+g_i)^t\hat{K}_t;\quad K_{it} = (1+g_i)^t\hat{K}_{it};\quad K_{ct} = (1+g_i)^t\hat{K}_{ct} \tag{1}$$
$$c_t = (1+g_c)^t\hat{c}_t \tag{2}$$

where the hat denote stationary variables.

Lets start with the capital goods sector. Since labor must be already stationary, we have:

$$I_t = (1+\gamma_i)^{t(1-v)}A_iK_{it}^vN_{it}^{1-v}$$
$$(1+g_i)^t\hat{I}_t = (1+\gamma_i)^{t(1-v)}A_i\left((1+g_i)^t\hat{K}_{it}\right)^vN_{it}^{1-v}$$
$$(1+g_i)^t\hat{I}_t = \left[(1+\gamma_i)^{1-v}(1+g_i)^v\right]^tA_i\hat{K}_{it}^vN_{it}^{1-v}$$
$$\hat{I}_t = \left[(1+\gamma_i)^{1-v}(1+g_i)^{v-1}\right]^tA_i\hat{K}_{it}^vN_{it}^{1-v}$$

This equation is stationary if

$$(1+\gamma_i)^{1-v}(1+g_i)^{v-1} = 1 \Rightarrow g_i = \gamma_i$$

Therefore,

$$\hat{I}_t = A_i\hat{K}_{it}^vN_{it}^{1-v}$$

Now, lets use the law of motion of capital to make sure the growth rate of capital is indeed g_i:

$$
\begin{aligned}
I_t &= K_{t+1} - (1-\delta)K_t \\
(1+g_i)^t \hat{I}_t &= (1+g_k)^{t+1}\hat{K}_{t+1} - (1-\delta)(1+g_k)^t\hat{K}_t \\
\hat{I}_t &= \left(\frac{1+g_k}{1+g_i}\right)^t (1+g_k)\hat{K}_{t+1} - (1-\delta)\left(\frac{1+g_k}{1+g_i}\right)^t \hat{K}_t
\end{aligned}
$$

This equation is stationary if

$$
\frac{1+g_k}{1+g_i} \Rightarrow g_i = g_k = \gamma_i
$$

Therefore,

$$
\hat{I}_t = (1+g_k)\hat{K}_{t+1} - (1-\delta)\hat{K}_t
$$

Note that, since both K_t and K_{it} are growing at the same rate, then the stock of capital in the consumption goods sector K_{ct} must be growing at the same rate as well. Thus conjecture (1) is correct.

Lets verify conjecture (2):

$$
\begin{aligned}
c_t &= (1+\gamma_c)^{t(1-\theta)} A_c K_{ct}^{\theta} N_{ct}^{1-\theta} \\
(1+g_c)^t \hat{c}_t &= (1+\gamma_c)^{t(1-\theta)} A_c \left((1+\gamma_i)^t \hat{K}_{ct}\right)^{\theta} N_{ct}^{1-\theta} \\
(1+g_c)^t \hat{c}_t &= \left[(1+\gamma_c)^{1-\theta}(1+\gamma_i)^{\theta}\right]^t A_c \hat{K}_{ct}^{\theta} N_{ct}^{1-\theta} \\
\hat{c}_t &= \left[(1+\gamma_i)^{\theta}(1+\gamma_c)^{1-\theta}(1+g_c)^{-1}\right]^t A_c \hat{K}_{ct}^{\theta} N_{ct}^{1-\theta}
\end{aligned}
$$

This equation is stationary if

$$
(1+g_c) = (1+\gamma_i)^{\theta}(1+\gamma_c)^{1-\theta} \Rightarrow g_c = (1+\gamma_i)^{\theta}(1+\gamma_c)^{1-\theta} - 1
$$

The growth rate of consumption is a weighted average of the technological progress in each sector. Finally, note that preferences also change when we detrend the model:

$$
\begin{aligned}
U &= \sum_{t=0}^{\infty} \beta^t (\log c_t - An_t) \\
&= \sum_{t=0}^{\infty} \beta^t (\log g_c^t \hat{c}_t - An_t) \\
&= \sum_{t=0}^{\infty} \beta^t (\log \hat{c}_t - An_t) + \sum_{t=0}^{\infty} (\beta g_c)^t
\end{aligned}
$$

Assuming that $\beta g_c < 1$ we have:

$$
U = \sum_{t=0}^{\infty} \beta^t (\log \hat{c}_t - An_t) + \frac{1}{1-\beta g_c}
$$

Without loss of generality we can ignore the constant term.

Planner's sequential problem:

$$\max \sum_{t=0}^{\infty} \beta^t [\log \hat{c}_t - A(N_{ct} + N_{it})] \tag{3}$$

$$\text{s.t. } \hat{c}_t = A_c \hat{K}_{ct}^{\theta} N_{ct}^{1-\theta}$$

$$\hat{I}_t = (1+\gamma_i)\left(\hat{K}_{ct+1} + \hat{K}_{it+1}\right) - (1-\delta)\left(\hat{K}_{ct} + \hat{K}_{it}\right) = A_i \hat{K}_{it}^{\nu} N_{it}^{1-\nu}$$

The FOCs are:

$$(1-\theta)A_c \hat{K}_{ct}^{\theta} N_{ct}^{-\theta} = (1-\nu)A_i \hat{K}_{it}^{\nu} N_{it}^{-\nu} \tag{3.1}$$

$$\theta A_c \hat{K}_{ct}^{\theta-1} N_{ct}^{1-\theta} = \nu A_i \hat{K}_{it}^{\nu-1} N_{it}^{1-\nu} \tag{3.2}$$

$$(1+\gamma_i)\frac{\hat{c}_{t+1}}{\hat{c}_t} = \beta\left(\nu A_i \hat{K}_{it+1}^{\nu-1} N_{it+1}^{1-\nu} + 1 - \delta\right) \tag{3.3}$$

Equations (3.1) and (3.2) show that the marginal products of labor and capital are equal across sectors. This must be true since there is no impediment to perfect mobility. Equation (3.3) is the usual capital euler equation.

In the decentralized economy, we have the following decision problems:

Household's sequential problem:

$$\max \sum_{t=0}^{\infty} \beta^t [\log \hat{c}_t - An_t] \tag{4}$$

$$\text{s.t. } \hat{c}_t = w_t n_t + r_t \hat{k}_t - q_t\left[(1+\gamma_i)\hat{k}_{t+1} - (1-\delta)\hat{k}_t\right]$$

Consumption goods producer:

$$\max_{\hat{K}_{ct}, N_{ct}} \left\{A_c \hat{K}_{ct}^{\theta} N_{ct}^{1-\theta} - w_t N_{ct} - r_t \hat{K}_{ct}\right\} \tag{5}$$

Capital goods producer:

$$\max_{\hat{K}_{it}, N_{it}} \left\{q_t A_i \hat{K}_{it}^{\nu} N_{it}^{1-\nu} - w_t N_{it} - r_t \hat{K}_{it}\right\} \tag{6}$$

Definition: a competitive equilibrium for this economy is an allocation $\left\{\hat{c}_t, n_t, \hat{k}_t, N_{ct}, N_{it}, \hat{K}_{ct}, \hat{K}_{it}\right\}_{t=0}^{\infty}$ and prices $\{w_t, r_t, q_t\}_{t=0}^{\infty}$ such that:

(i) Given prices, the household solves problem (4)

(ii) Given prices, the consumption goods producer solves problem (5)

(iii) Given prices, the capital goods producer solves problem (6)

(iv) Markets clear:

$$\begin{aligned}
\hat{k}_t &= \hat{K}_{it} + \hat{K}_{ct} \\
n_t &= N_{it} + N_{ct} \\
A_c \hat{K}_{ct}^{\theta} N_{ct}^{1-\theta} &= \hat{c}_t \\
A_i \hat{K}_{it}^{\nu} N_{it}^{1-\nu} &= (1+\gamma_i)\hat{k}_{t+1} - (1-\delta)\hat{k}_t
\end{aligned}$$

Note: this definition is slightly different from the recursive competitive equilibrium that we are used to.

(b) There exists a steady state even if consumption and investment grow at different rates. Since there are no distortions, it is easier to analyze the steady state by looking at the planner's solution. The equations that characterize the planner's solution in steady state are:

$$\begin{aligned}
\frac{1-\theta}{\theta}\frac{\hat{K}_c}{N_c} &= \frac{1-\nu}{\nu}\frac{\hat{K}_i}{N_i} \\
1+\gamma_i &= \beta\left(\nu A_i \hat{K}_i^{\nu-1} N_i^{1-\nu} + 1 - \delta\right) \\
\hat{c} &= A_c \hat{K}_c^{\theta} N_c^{1-\theta} \\
A_i \hat{K}_i^{\nu} N_i^{1-\nu} &= (\gamma_i + \delta)\left(\hat{K}_c + K_i\right) \\
\bar{N} &= N_c + N_i
\end{aligned}$$

where the first equation is just the combination of (3.1) and (3.2), the second is the euler equation, and the remaining are just the resource constraints. These 5 equations determine the 5 unknowns: $c, N_c, N_i, \hat{K}_c, \hat{K}_i$. (If you work on the algebra a bit you can get the explicit solutions for each variable).

(c) Given that marginal products are equal to input prices in both sectors, the FOCs for the firms imply:

$$\begin{aligned}
q\nu A_i \hat{K}_i^{\nu-1} N_i^{1-\nu} &= r = \theta A_c \hat{K}_c^{\theta-1} N_c^{1-\theta} \\
q(1-\nu) A_i \hat{K}_i^{\nu} N_i^{-\nu} &= w = (1-\theta) A_c \hat{K}_c^{\theta} N_c^{-\theta}
\end{aligned}$$

Therefore,

$$q = \frac{\theta A_c \hat{K}_c^{\theta-1} N_c^{1-\theta}}{\nu A_i \hat{K}_i^{\nu-1} N_i^{1-\nu}} = \frac{(1-\theta) A_c \hat{K}_c^{\theta} N_c^{-\theta}}{(1-\nu) A_i \hat{K}_i^{\nu} N_i^{-\nu}} \tag{7}$$

(d) Since the ratio of marginal products (7) is not affected by the tax rate, I guess the price of capital will not change in the long run.

(e) Suppose the tax is imposed in period T. From this period on the household's budget constraint changes to:

$$(1+\tau)\hat{c}_t = w_t n_t + r_t \hat{k}_t - q_t\left[(1+\gamma_i)\hat{k}_{t+1} - (1-\delta)\hat{k}_t\right] + R_t, \forall t \geq T$$

where τ is the consumption tax rate and R is the tax rebate.

To gain intuition, let λ_t be the lagrange multiplier on the houshold's budget constraint. The FOC for consumption in period $t \geq T$ is:

$$u_{ct} = \lambda_t(1+\tau)$$

The left-hand side is just the marginal utility of consumption. The right-hand side is the shadow price of consumption. This shadow price increases with the tax, which discourages consumption. Try to figure out what happens with the other variables.

SPRING 2002 - QUESTION 1

(a)

Bellman equation:

$$v(k_{-1},K,J) = \max_k \left\{ pfk - J(k-k_{-1}) - \frac{d}{2}(k-k_{-1})^2 + \beta E[v(k,K',J')] \right\} \quad (1)$$

$$\begin{aligned} \text{s.t. } J' &= \gamma J + \varepsilon' \\ p &= a - bfK + u \\ K' &= G(K,J',u') \\ u' &= i.i.d \text{ with zero mean} \end{aligned} \quad \#$$

(b) A Recursive competitive equilibrium for this industry is:

(i) A policy function $k = k(k_{-1},K,J,u)$

(ii) Exogenous prices p and J

(iii) An industry wide stock of capital $G(K,J',u')$

such that

(1) Given (ii) firms solve problem 1

(2) Capital market clears:

$$K = Nk$$

(3) Firm's predictions are correct:

$$G(K,J',u') = Nk'$$

(c) The FOC with respect to k is:

$$J + d(k-k_{-1}) = pf + \beta E[v_1(k,K',J')]$$

The envelope conditions is:

$$v_1(k_{-1},K,J) = J + d(k-k_{-1}) \Rightarrow v_1(k,K',J') = J' + d(k'-k)$$

Substituting the EC into the FOC gives:

$$\begin{aligned} J + d(k - k_{-1}) &= pf + \beta E[J' + d(k' - k)] \\ &= pf + \beta\gamma J + d(E[k'] - k) \end{aligned}$$

Now use the equilibrium condition $K = Nk$ to obtain:

$$J + d\left(\frac{K}{N} - \frac{K_{-1}}{N}\right) = pf + \beta\gamma J + d\left(E\left[\frac{K'}{N}\right] - \frac{K}{N}\right)$$

After rearranging terms and substituting for p we get the following euler equation:

$$E_t\left[K_{t+1} - \left(2 + \frac{Nbf^2}{d}\right)K_t + K_{t-1} - (1 - \beta\gamma)\frac{N}{d}J_t + \frac{Nf}{d}u_t + a\frac{Nf}{d}\right] = 0 \tag{2}$$

This is a linear stochastic second-order difference equation.

To characterize the solution, a reasonable conjecture would be

$$K_t = \alpha_0 + \alpha_1 K_{t-1} + \alpha_2 J_t + \alpha_3 u_t \tag{3}$$

Substitute the conjecture (3) and the law of motion of J in (2) to get:

$$\begin{aligned} 0 = &\left[\alpha_0\left(\alpha_1 - 1 - \frac{Nbf^2}{d}\right) + a\frac{Nf}{d}\right] + \left[1 + \alpha_1\left(\alpha_1 - 2 - \frac{Nbf^2}{d}\right)\right]K_{t-1} + \\ &\left[\alpha_2\left(\alpha_1 + \gamma - 2 - \frac{Nbf^2}{d}\right) - (1 - \beta\gamma)\frac{N}{d}\right]J_t + \left[\frac{Nf}{d} + \alpha_3\left(\alpha_1 - 2 - \frac{Nbf^2}{d}\right)\right]u_t \end{aligned}$$

By MUC, a solution to this equation must satisfy

$$\alpha_0\left(\alpha_1 - 1 - \frac{Nbf^2}{d}\right) + a\frac{Nf}{d} = 0 \tag{4.1}$$

$$1 + \alpha_1^2 - \alpha_1\left(2 + \frac{Nbf^2}{d}\right) = 0 \tag{4.2}$$

$$\alpha_2\left(\alpha_1 + \gamma - 2 - \frac{Nbf^2}{d}\right) - (1 - \beta\gamma)\frac{N}{d} = 0 \tag{4.3}$$

$$\frac{Nf}{d} + \alpha_3\left(\alpha_1 - 2 - \frac{Nbf^2}{d}\right) = 0 \tag{4.4}$$

Use (4.2) to solve for α_1:

$$\begin{aligned} \alpha_1 &= \frac{1}{2}\left[2 + \frac{Nbf^2}{d} \pm \sqrt{\left(2 + \frac{Nbf^2}{d}\right)^2 - 4}\right] \\ &= 1 + \frac{Nbf^2}{2d} \pm \sqrt{\left(1 + \frac{Nbf^2}{2d}\right)^2 - 1} \end{aligned}$$

We can verify that both solutions are real and positive. Additionally, one solution is larger than one and the other is smaller than one. To satisfy the TVC pick up the solution smaller than one:

$$\alpha_1 = 1 + \frac{Nbf^2}{2d} - \sqrt{\left(1 + \frac{Nbf^2}{2d}\right)^2 - 1}$$

Given this solution for α_1, we can back up the other parameters:

$$\alpha_0 = \frac{aNf}{d(1-\alpha_1) + Nbf^2} > 0$$
$$\alpha_2 = \frac{-(1-\beta\gamma)N}{d(2-\alpha_1-\gamma) + Nbf^2} < 0$$
$$\alpha_3 = \frac{Nf}{d(2-\alpha_1) + Nbf^2} > 0$$

Note that the sign of these last three parameters make economic sense. First, we can interpret α_0 as the average or the long-run stock of capital, which is positive. Second, α_2 implies a negative relationship between the price of capital and the demand for capital, as expected. Finally, α_3 implies that the capital stock is positively affected by demand shocks, that is, when the price of output goes up the demand for capital also goes up, and vice-versa.

SPRING 2002 - QUESTION 2

(a) First, get the per capita variables. Using the resource constraint:

$$c_t + (1+\eta)k_{t+1} = \gamma^t k_t^\theta l_t^{1-\theta-\phi}$$

The small letters denote per capita variables. Note that $l_t = 1/(1+\eta)^t$, then

$$c_t + (1+\eta)k_{t+1} = \Gamma_t k_t^\theta$$

where $\Gamma_t = [\gamma(1+\eta)^{\theta+\phi-1}]^t$.

Also, note that the discount factor becomes $\beta(1+\eta)$. We need the condition that $\beta(1+\eta) < 1$.

The Bellman equation for the social planners is:

$$v(k,\Gamma) = \max_{K'}\{\log(\Gamma k^\theta - (1+\eta)k') + \beta(1+\eta)v(k',\Gamma')\}$$
$$\text{s.t. } \Gamma' = [\gamma(1+\eta)^{\theta+\phi-1}]\Gamma$$

(b) Along the BGP all variables grow at the same and constant rate g.

$$c_t = g^t \hat{c}_t$$
$$k_t = g^t \hat{k}_t$$
$$y_t = g^t \hat{y}_t$$

The hat variables denote the BGP levels of consumption, capital and output. These variables are stationary and have a well defined steady state. Use the resource constraint to pin down the growth rate:

$$g^t \hat{c}_t + (1+\eta) g^{t+1} \hat{k}_{t+1} = \Gamma_t \left(g^t \hat{k}_t\right)^{\theta}$$
$$\hat{c}_t + (1+\eta) g \hat{k}_{t+1} = (\Gamma_t g^{(\theta-1)t}) \hat{k}_t^{\theta}$$

This is stationary if

$$\Gamma_t g^{\theta t} = 1 \Leftrightarrow [\gamma(1+\eta)^{\theta+\phi-1} g^{\theta-1}]^t = 1 \Leftrightarrow g = \left(\frac{\gamma}{(1+\eta)^{1-\theta-\phi}} \right)^{\frac{1}{1-\theta}}$$

Therefore, along the BGP consumption grows according to

$$\frac{c_{t+1}}{c_t} = \left(\frac{\gamma}{(1+\eta)^{1-\theta-\phi}} \right)^{\frac{1}{1-\theta}}$$

There is an equivalent way to pin down the growth rate of consumption. Remember that CRRA utility functions are consistent with BGP. The log utility is just a special case of CRRA. This means that we can use the euler equation to pin down g. The euler equation for the planner's problem is:

$$\frac{c_{t+1}}{c_t} = \beta\theta\Gamma_{t+1} k_{t+1}^{\theta-1}$$

Along the steady state BGP we have $c_t = g^t \bar{c}$ and $k_t = g^t \bar{k}$. Hence,

$$g = \beta\theta(\Gamma_{t+1} g^{(\theta-1)t+1}) \bar{k}^{\theta-1}$$

The LHS is already stationary. The RHS is stationary if $\Gamma_{t+1} g^{(\theta-1)t+1} = 1$, which gives the same growth rate we got before.

(c) Let q be the price of land, and let v, w and r be the rental rate of land, labor and capital, respectively. I will define the equilibrium in terms of each household member. You could also do it for the entire household.

The household budget constraint is in per capita terms is:

$$c_t + k_{t+1} + q_t[l_{t+1} - l_t] = w_t + r_t k_t + v_t l_t$$

The term $l_{t+1} - l_t$ denotes net investment on land (land purchases).

The Bellman equation for each household member is:

$$v(k,l,K,\Gamma,N) = \max_{k',l'}\{\log c + \beta v(k',l',K',\Gamma',N')\} \tag{1}$$

$$\begin{aligned} \text{s.t. } c &= w(K,\Gamma,N) + r(K,\Gamma,N)k + v(K,\Gamma,N)l - k' - q(K,\Gamma,N)l' \\ K' &= G(K,\Gamma,N) \\ \Gamma' &= \gamma\Gamma \\ N' &= (1+\eta)N \end{aligned}$$

The firm's problem is:

$$\max_{K^f,N^f,L^f}\left\{\Gamma(K^f)^{\theta}(N^f)^{\phi}(L^f)^{1-\theta-\phi} - v(K,\Gamma,N)L^f - w(K,\Gamma,N)N^f - r(K,\Gamma,N)K^f\right\} \tag{2}$$

A recursive competitive equilibrium for this economy is:

(1) A set of policy functions $k'(k,l,K,\Gamma,N)$, $l'(k,l,K,\Gamma,N)$ and $c(k,l,K,\Gamma,N)$ for each member of the household

(2) A set of policy functions for the firm $K^f(K,\Gamma,N), N^f(K,\Gamma,N)$ and $L^f(K,\Gamma,N)$

(3) Princing functions $v(K,\Gamma,N), w(K,\Gamma,N), r(K,\Gamma,N)$ and $q(K,\Gamma,N)$

(4) A law of motion for the aggregate endogenous state $K' = G(K,\Gamma,N)$

(5) A law of motion for the aggregate exogenous state $\Gamma' = \gamma\Gamma$

such that

(i) Given (3), (4), and (5) each member of the household solves problem (1)

(ii) Given (3), (4), and (5) the firm solves problem (2)

(iii) Markets clear (for completeness I list all markets below):

$$\begin{aligned} \text{Labor Services} &: N^f = N \\ \text{Land Services} &: L^f = Nl\left(K,\frac{1}{N},K,\Gamma,N\right) = 1 \\ \text{Capital Services} &: K^f = Nk\left(K,\frac{1}{N},K,\Gamma,N\right) = K \\ \text{Land} &: l'\left(K,\frac{1}{N},K,\Gamma,N\right) - l\left(K,\frac{1}{N},K,\Gamma,N\right) = -\frac{\eta}{(1+\eta)N} \\ \text{Output} &: Nc\left(K,\frac{1}{N},K,\Gamma,N\right) + G(K,\Gamma) = \Gamma(K^f(K,\Gamma,N))^{\theta}(N)^{\phi} \end{aligned}$$

(iv) Aggregate consistency:

$$\begin{aligned} k'\left(K,\frac{1}{N},K,\Gamma,N\right) &= G(K,\Gamma,N) \\ l'\left(K,\frac{1}{N},K,\Gamma,N\right) &= \frac{1}{(1+\eta)N} \end{aligned}$$

SPRING 2002 - QUESTION 6

(a) I interpret the parameter ϕ as a preference shock, in particular a shock to leisure (or to the supply of labor).

(b) To characterize the recursive competitive equilibrium let us first write down the Bellam equation for the representative consumer:

$$v(K,k,\phi) = \max_{l,i}\left\{w(K,\phi)l + r(K,\phi)k - i' - \phi\frac{l^2}{2} + \beta E[v((1-\delta)K + I, (1-\delta)k + i, \phi')]\right\} \quad (1)$$

s.t. $I = I(K,\phi)$

$\phi' > 0$ is a random shock with some probability distribution

Note: Be careful when preferences are linear or quasi-linear. In the present case, preferences are linear in consumption, which means that there can be corner solutions. Let $R' = r' + 1 - \delta$ denote the return on capital between today and tomorrow, and let $MRS_{c,c'} = 1/\beta$ denote the marginal rate of substitution between consumption today and consumption tomorrow. Note that consumption today and consumption tomorrow are perfect substitutes (linear indifference curves). Then the following three cases are possible:

$$MRS_{c,c'} > R' \Rightarrow c = wl + rk \Rightarrow i = 0$$
$$MRS_{c,c'} < R' \Rightarrow c = 0 \Rightarrow i = wl + rk$$
$$MRS_{c,c'} = R' \Rightarrow c \in [0, wl + rk] \Rightarrow i \in [wl + rk, 0]$$

The firm's problem is:

$$\max_{k^f,l^f}\left\{A(k^f)^{\theta}(l^f)^{1-\theta} - w(K,\phi)l^f - r(K,\phi)k^f\right\} \quad (2)$$

A recursive competitive equilibrium for this economy is:

(1) A set of policy functions $l(K,k,\phi)$ and $i(K,k,\phi)$ for the household

(2) A set of policy functions $l^f(K,\phi)$ and $k^f(K,\phi)$ for the firm

(3) Pricing functions $w(K,\phi)$ and $r(K,\phi)$

(4) An aggregate decision rule $I(K,\phi)$

such that

(i) Given (3) and (4) the household solves problem (1)

(ii) Given (3) the firm solves problem (2)

(iii) Markets clear:

$$l^f(K,\phi) = l(K,K,\phi)$$
$$k^f(K,\phi) = k(K,K,\phi) = K$$

(iv) The predictions are correct:

$$i(K,K,\phi) = I(K,\phi) = \begin{cases} 0 & \text{if } \frac{1}{\beta} > r(K',\phi') + 1 - \delta \\ A(k^f(K,\phi))^{\theta}(l^f(K,\phi))^{1-\theta} & \text{if } \frac{1}{\beta} < r(K',\phi') + 1 - \delta \\ \in \left[0, A(k^f(K,\phi))^{\theta}(l^f(K,\phi))^{1-\theta}\right] & \text{if } \frac{1}{\beta} = r(K',\phi') + 1 - \delta \end{cases}$$

(c) If ϕ is constant and if $\frac{1}{\beta} = R'$ the economy could achieve a positive steady state. The steady state capital euler equation in this case is:

$$\frac{1}{\beta} = \theta A\left(\frac{\bar{l}}{\bar{k}}\right)^{1-\theta} + 1 - \delta \tag{3}$$

The labor euler equation is:

$$\phi\bar{l} = (1-\theta)A\left(\frac{\bar{k}}{\bar{l}}\right)^{\theta} \tag{4}$$

Use (3) and (4) to solve for $\bar{k}$ and $\bar{l}$:

$$\bar{l} = \frac{(1-\theta)A}{\phi}\left(\frac{A\beta\theta}{1-\beta(1-\delta)}\right)^{\frac{\theta}{1-\theta}}$$

$$\bar{k} = \frac{(1-\theta)A}{\phi}\left(\frac{A\beta\theta}{1-\beta(1-\delta)}\right)^{\frac{1+\theta}{1-\theta}}$$

(d) If $k_0 < \bar{k} \Rightarrow \frac{1}{\beta} < \bar{R} \Rightarrow$ all resources will be invested until $\frac{1}{\beta} = \bar{R}$. On the other hand, if $k_0 > \bar{k} \Rightarrow \frac{1}{\beta} > \bar{R} \Rightarrow$ the economy will consume part of the available capital until $\frac{1}{\beta} = \bar{R}$. If investment can go negative, the convergence will be potentially slower in the first case and faster in the second case. See what happens with the competitive equilibrium.

(e) Consider again the case of a constant ϕ and $\frac{1}{\beta} = R'$. Now consider all the equations that characterize the planner's solution:

$$\text{Capital euler equation}: \frac{1}{\beta} = \theta A k_{t+1}^{\theta-1}(x_{t+1}l_{t+1})^{1-\theta} + 1 - \delta$$

$$\text{Labor euler equation}: \phi l_t = (1-\theta)Ak_t^{\theta}(x_t l_t)^{-\theta} + 1 - \delta$$

$$\text{Resource constraint}: c_t + k_{t+1} = Ak_t^{\theta}(x_t l_t)^{1-\theta} + (1-\delta)k_t$$

Lets conjecture that along the BGP all variables grow at the same and constant rate g. Now lets check if the above three equations satisfy this BGP. Using $y_t = g^t\bar{y}$ for each variable y we get:

$$\frac{1}{\beta} = \theta A\bar{k}^{\theta-1}\bar{l}^{1-\theta}[g^{\theta-1}(1+\gamma)^{1-\theta}]^{t+1} + 1 - \delta$$

$$\phi\bar{l} = (1-\theta)A\bar{k}^{\theta}\bar{l}^{-\theta}[g^{\theta}(1+\gamma)^{-\theta}]^{t} + 1 - \delta$$

$$\bar{c} + g\bar{k} = A\bar{k}^{\theta}\bar{l}^{1-\theta}[g^{\theta-1}(1+\gamma)^{1-\theta}]^{t} + (1-\delta)\bar{k}$$

The same growth rate makes all equations stationary. The growth rate is $g = 1+\gamma$. Therefore, in the steady state growth path that we conjectured we have:

$$\frac{1}{\beta} = \theta A\bar{k}^{\theta-1}\bar{l}^{1-\theta} + 1 - \delta$$

$$\phi\bar{l} = (1-\theta)A\bar{k}^{\theta}\bar{l}^{-\theta} + 1 - \delta$$

$$\bar{c} + (1+\gamma)\bar{k} = A\bar{k}^{\theta}\bar{l}^{1-\theta} + (1-\delta)\bar{k}$$

You can solve for $\bar{k}, \bar{c}, \bar{l}$.

Remember from Hansen's lecture that CRRA utility functions are consistent with BGP. Are linear preferences a special case of CRRA?

SPRING 2003 - QUESTION 2

(a)

Bellman equation:

$$v(k) = \max_{k'}\left\{\frac{(Ak-k')^{1-\sigma}}{1-\sigma} + \beta v(k')\right\}$$

A sequence-of-markets equilibrium for this economy is an allocation $\{c_t, k_t\}_{t=0}^{\infty}$ and a sequence of prices $\{r_t\}_{t=0}^{\infty}$ such that

(i) Given prices, consumers solve:

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

s. t. $c_t + k_{t+1} = r_t k_t, \quad \forall t$

k_0 is given

(i) Given prices, firms solve:

$$\max_{k_t^f}\left\{Ak_t^f - r_t k_t^f\right\}, \quad \forall t$$

(iii) Markets clear:

$$k_t = k_t^f, \forall t$$

$$c_t + k_{t+1} = Ak_t, \forall t$$

A balanced growth path (BGP) for this economy is a growth rate g such that in the **long run** all variables grow at this constant rate:

$$\frac{k_{t+1}}{k_t} = g$$
$$\frac{c_{t+1}}{c_t} = g$$
$$\frac{y_{t+1}}{y_t} = g$$

To get g, use the above Bellman equation, compute FOC and EC, and combine them in order to get the following Euler equation:

$$(Ak_t - k_{t+1})^{-\sigma} = \beta A(Ak_{t+1} - k_{t+2})^{-\sigma}$$

Rearranging terms gives:

$$\frac{c_{t+1}}{c_t} = \frac{Ak_{t+1} - k_{t+2}}{Ak_t - k_{t+1}} = (A\beta)^{\frac{1}{\sigma}} \tag{1}$$

Therefore, along the **long-run** balanced growth path we have:

$$g = (A\beta)^{\frac{1}{\sigma}}$$

For g to be a BGP growth rate we require $A\beta > 1$.

To study whether the BGP is reached instantaneously, consider (1) again:

$$\frac{Ak_{t+1} - k_{t+2}}{Ak_t - k_{t+1}} = (A\beta)^{\frac{1}{\sigma}}$$

Divide numerator and denominator of the left-hand side by k_t and obtain:

$$\frac{A\frac{k_{t+1}}{k_t} - \frac{k_{t+2}}{k_t}\frac{k_{t+1}}{k_{t+1}}}{A\frac{k_t}{k_t} - \frac{k_{t+1}}{k_t}} = (A\beta)^{\frac{1}{\sigma}}$$

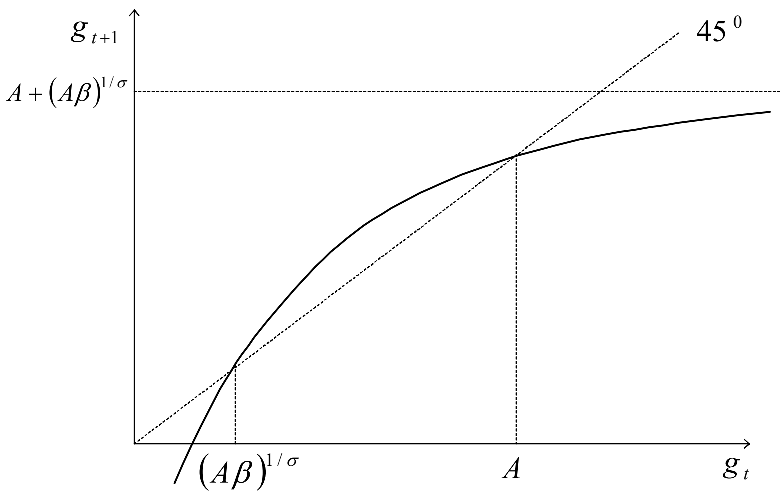
Given that in the short-run (or at date $t = 0$), before reaching the BGP, the economy can potentially grow at a rate that is not constant implies:

$$\frac{Ag_t - g_{t+1}g_t}{A - g_t} = (A\beta)^{\frac{1}{\sigma}}$$

where $g_t = k_{t+1}/k_t$ is not constant in principle but could vary over time. Rearranging the above equation we get:

$$g_{t+1} = A - (A\beta)^{\frac{1}{\sigma}}\left(\frac{A}{g_t} - 1\right) \tag{2}$$

It is easy to check that (2) is strictly concave and converges to two possible steady states, A and $(A\beta)^{\frac{1}{\sigma}}$, depending on the initial condition. As the next figure shows, A is a stable steady state and $(A\beta)^{\frac{1}{\sigma}}$ is unstable. Additionally, it is easy to check that A does not satisfy the TVC, hence we can rule it out. Therefore, in the long-run the economy must be growing at the rate $(A\beta)^{\frac{1}{\sigma}}$ as suggested above. Since this steady state growth rate is unstable, the economy has to reach it instantaneously, otherwise there would be no BGP in the long-run (the economy would diverge).



(b)

In order to write down the Bellman equation realize that $c = Ak - k'$ and $c_{-1} = Ak_{-1} - k$. There will be two state variables k and k_{-1}:

$$v(k_{-1},k) = \max_{k'}\left\{ \frac{[A(k-\phi k_{-1}) - (k'-\phi k)]^{1-\sigma}}{1-\sigma} + \beta v(k,k') \right\}$$

Combining FOC and ECs gives the following Euler equation:

$$\frac{1}{(c_t - \phi c_{t-1})^\sigma} = \frac{\beta(A+\phi)}{(c_{t+1}-\phi c_t)^\sigma} - \frac{\phi\beta^2 A}{(c_{t+2}-\phi c_{t+1})^\sigma} \qquad (3)$$

Multiply both sides of (3) by $(c_{t+1}-\phi c_t)^\sigma$ to get:

$$\left(\frac{c_{t+1}-\phi c_t}{c_t-\phi c_{t-1}}\right)^\sigma = \beta(A+\phi) - \phi\beta^2 A\left(\frac{c_{t+1}-\phi c_t}{c_{t+2}-\phi c_{t+1}}\right)^\sigma \qquad (4)$$

Use the fact that $\frac{c_{t+1}}{c_t} = g_t$ and rewrite (4) as follows:

$$\left(\frac{g_t c_t - \phi c_t}{c_t - \phi\frac{c_t}{g_{t-1}}}\right)^\sigma = \beta(A+\phi) - \phi\beta^2 A\left(\frac{g_t c_t - \phi c_t}{g_t g_{t+1} c_t - \phi g_t c_t}\right)^\sigma$$

This simplifies to

$$\left(g_{t-1}\frac{g_t-\phi}{g_{t-1}-\phi}\right)^{\sigma} = \beta(A+\phi) - \phi\beta^2 A\left(\frac{1}{g_t}\frac{g_t-\phi}{g_{t+1}-\phi}\right)^{\sigma} \tag{5}$$

Along the BGP we have $g_{t-1} = g_t = g_{t+1} = g, \forall t$. Hence, (5) becomes:

$$g^{\sigma} = \beta(A+\phi) - \frac{\phi\beta^2 A}{g^{\sigma}} \tag{6}$$

Define $\hat{g} \equiv g^{\sigma}$ and rewrite (6) as follows:

$$\hat{g}^2 - \beta(A+\phi)\hat{g} + \phi A\beta^2 = 0$$

Solving for $\hat{g}$ we obtain:

$$\hat{g} = \begin{cases} \hat{g}_H = A\beta \\ \hat{g}_L = \phi\beta \end{cases}$$

Hence,

$$g = \begin{cases} g_H = (A\beta)^{\frac{1}{\sigma}} \\ g_L = (\phi\beta)^{\frac{1}{\sigma}} \end{cases} \tag{7}$$

Note that g_L cannot be a BGP growth rate because $g_L < 1$, which implies that the stock of capital goes to zero as $t \to \infty$. Therefore, we can rule g_L out. Like before we require $A\beta > 1$ so that $g_H > 1$.

My guess is that the economy does not have to reach the BGP immediately because of the habit formation preferences. What do you think?

FALL 2003 - QUESTION 1

The Solow growth model without technological progress and population growth is described by the following equations (all variables in per capita terms):

$$\begin{aligned} \text{Production function} &: y_t = f(k_t) \\ \text{Savings function} &: s_t = sy_t = sf(k_t) \\ \text{Law of motion of capital} &: k_{t+1} = (1-\delta)k_t + i_t \\ \text{Resource constraint} &: y_t = c_t + i_t \end{aligned}$$

where $f(.)$ satisfies the Inada conditions, and $s \in (0,1)$ is the exogenous savings rate.

In equilibrium we have

$$s_t = i_t = sf(k_t)$$

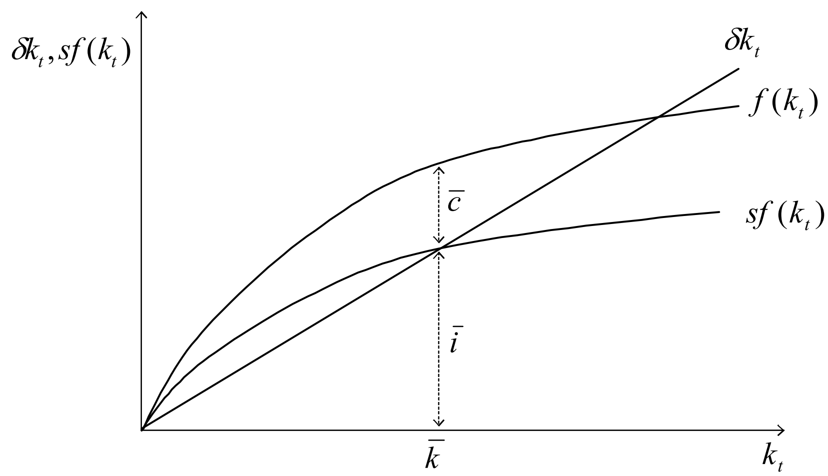
Using this conditions and the law of motion of capital we get:

$$k_{t+1} - k_t = \Delta k_t = sf(k_t) - \delta k_t$$

In steady state $\Delta k_t = 0$ and savings are just enough to recoup the capital that wears out every period:

$$sf(\bar{k}) = \delta \bar{k}$$

The following graph illustrates the long-run equilibrium, which is stable:



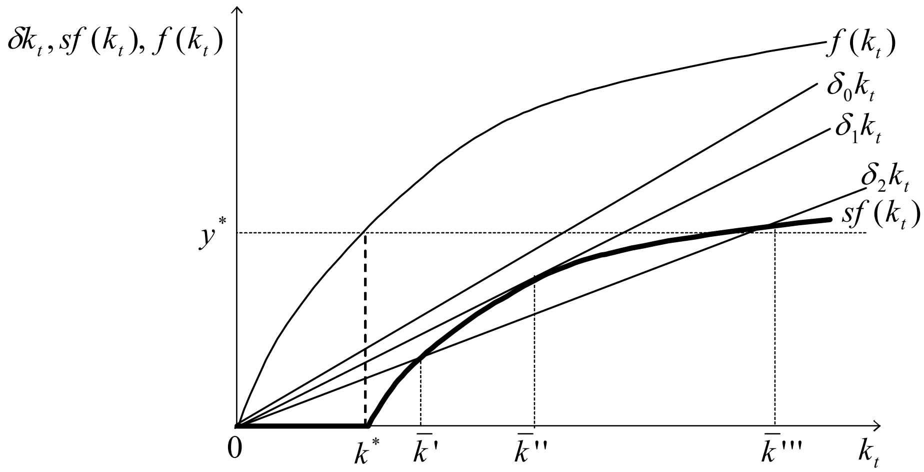
(a) In the Solow growth model with subsistence consumption, the savings function changes as follows:

$$s_t = \begin{cases} s(y - y^*) & \text{if } y > y^* \\ 0 & \text{if } y \leq y^* \end{cases}$$

Now, the capital accumulation is given by:

$$\Delta k_t = \begin{cases} sf(k_t) - sy^* - \delta k_t & \text{if } y > y^* \\ -\delta k_t & \text{if } y \leq y^* \end{cases}$$

There are several steady state candidates, depending on k^*, y^* and δ. The next figure illustrates some of the candidates.



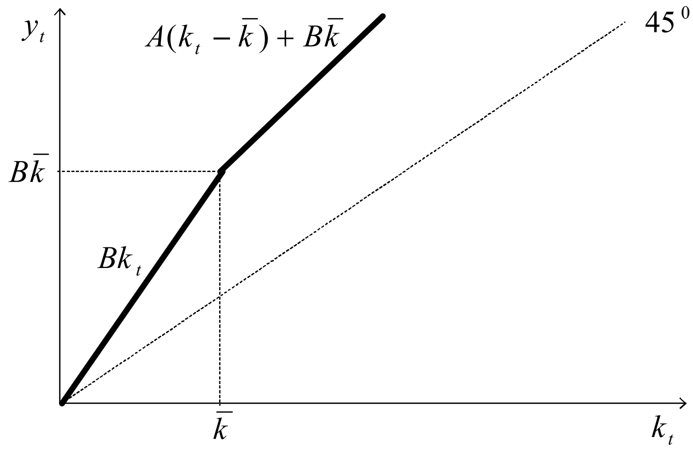
If k^* is low enough, there could be multiple positive steady states, unique positive steady state, or no positive steady state at all. For example, if the depreciation rate is very high (δ_0), the only possible steady state is $\bar{k} = 0$ (stable). In this case, the economy is unable to save enough to keep up the stock of capital and goes to a sort of "poverty trap" in the long-run. If the depreciation rate is not very high (δ_1), there could be a unique steady state $\bar{k}''$, which is stable from above and unstable from below. Finally, if the depreciation rate is very small (δ_2), there could be multiple steady states. In particular the economy could be in a low steady state $(\bar{k}')$, which is unstable, or in a high steady state $(\bar{k}''')$, which is stable.

If k^* is very high, there could be no depreciation rate low enough such that the economy reaches a positive steady state.

(b) Economies that start with a very low per capita capital may not sustain savings and investment in face of the required subsistence consumption. For example, if $\bar{k}' < k_0 < \bar{k}''$, the economy will converge to $\bar{k}'$ in the long run (this is a sort of poverty trap with positive stock of capital). It is also possible that the economy starts with an even lower stock of capital, say $k_0 < \bar{k}'$. In this case, the only feasible stock of capital in the long run is zero.

FALL 2003 - QUESTION 2

(a) Draw the production function



(b) Sequence-of-markets equilibrium

A sequence-of-markets equilibrium for this economy is an allocation $\{c_t, k_t\}_{t=0}^{\infty}$ and price $\{r_t\}_{t=0}^{\infty}$ such that

(i) Given prices, consumers solve:

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t c_t$$

such that

$$c_t + k_{t+1} = r_t k_t, \quad \forall t$$

k_0 is given

(i) Given prices, firms solve:

$$\max \begin{cases} Bk_t^f - r_t k_t^f & \text{if } k_t^f < \bar{k}^f \\ A(k_t - \bar{k}^f) + B\bar{k}^f - r_t k_t^f & \text{if } k_t^f \geq \bar{k}^f \end{cases}$$

Note that the marginal product of capital decreases as the production function changes:

$$r_t = \begin{cases} B & \text{if } k_t^f < \bar{k}^f \\ A < B & \text{if } k_t^f \geq \bar{k}^f \end{cases}$$

(iii) Markets clear:

$$k_t^f = k_t, \forall t$$

$$c_t + k_{t+1} = y_t, \forall t$$

(c) Bellman equation:

$$v(k) = \begin{cases} \max_{k'}\{Bk - k' + \beta v(k')\} & \text{if } k < \bar{k} \\ \max_{k'}\{A(k-\bar{k}) + B\bar{k} - k' + \beta v(k')\} & \text{if } k \geq \bar{k} \end{cases}$$

(d) The steady state capital is $k_{ss} = 0$. To see why, first compute the marginal rate of substitution (MRS) between c_t and c_{t+1} and compare with the marginal return or marginal product (MP) of capital:

$$MRS_{c_t,c_{t+1}} = \frac{1}{\beta}\begin{cases} < B & \text{if } k_t < \bar{k} \\ > A & \text{if } k_t \geq \bar{k} \end{cases}$$

Now consider the only two possible scenarios:

(1) $0 < k_0 < \bar{k} \Rightarrow MRS < MP \Rightarrow$ the consumer prefers to consume zero in each period and accumulate capital until $k_t = \bar{k}$. When the economy reaches $\bar{k}$ we have $MRS > MP$ so the consumer prefers to consume all accumulated capital and invest zero from that period on.

(2) $k_0 \geq \bar{k} \Rightarrow MRS > MP \Rightarrow$ the consumer prefers to consume all accumulated capital in the first period and invest zero afterward.

Therefore, the consumer eats the stock of capital either in the future or today. In both cases, the stock of capital is zero in the long run.

(e) Optimal law of motion

$$k_{t+1} = \begin{cases} Bk_t & \text{if } k_t < \bar{k} \\ 0 & \text{if } k_t \geq \bar{k} \end{cases}$$

(e) Value function. Assuming the economy reaches $\bar{k}$ in finite time (say T periods), if it starts with $0 < k_0 < \bar{k}$:

$$v(k_0) = \begin{cases} (\beta B)^{T-1}Bk_0 & \text{if } k_0 < \bar{k} \\ Bk_0 & \text{if } k_0 \geq \bar{k} \end{cases}$$

FALL 2003 - QUESTION 4

(a) First, we need to detrend all variables. Conjecture that all variables grow at the same rate along the balanced growth path, say $x_t = \hat{x}_t g^t$, where $\hat{x}_t$ denotes the detrended variables. Start with the resource constraint:

$$g^t\hat{c}_t + g^{t+1}\hat{K}_{t+1} = z_t(1+\gamma)^{t(1-\theta)}\left(\hat{K}_t g^t\right)^{\theta} N_t^{1-\theta} + (1-\delta)\hat{K}_t g^t$$

Rearranging terms gives:

$$\hat{c}_t + g\hat{K}_{t+1} = z_t\left(\frac{(1+\gamma)^{(1-\theta)}}{g^{1-\theta}}\right)^t \hat{K}_t^{\theta}N_t^{1-\theta} + (1-\delta)\hat{K}_t$$

The problem is stationary if $g = 1+\gamma$. In this case,

$$\hat{c}_t + (1+\gamma)\hat{K}_{t+1} = z_t\hat{K}_t^{\theta}N_t^{1-\theta} + (1-\delta)\hat{K}_t$$

Now, adjust the lifetime utility:

$$\begin{aligned}
&\max E_0 \sum_{t=0}^{\infty} \beta^t[\log(\hat{c}_t(1+\gamma)^t) - v(n_t)] \\
&= \max E_0 \sum_{t=0}^{\infty} \beta^t[\log(\hat{c}_t) - v(n_t)] + \sum_{t=0}^{\infty} \beta^t t\log(1+\gamma) \\
&= \max E_0 \sum_{t=0}^{\infty} \beta^t[\log(\hat{c}_t) - v(n_t)] + \frac{\beta\log(1+\gamma)}{(1-\beta)^2} \\
&= \max E_0 \sum_{t=0}^{\infty} \beta^t[\log(\hat{c}_t) - v(n_t)] + constant
\end{aligned}$$

From now on we can ignore the constant above. Additionally, and without loss of generality, suppose the population have a constant unity mass, which implies $n = N, \forall t$.

The Bellman equation for the planner's problem is:

$$\begin{aligned}
V(z,\hat{K}) &= \max_{\hat{K}',n}\left\{\log\hat{c} - v(n) + \beta E\left[V\left(z',\hat{K}'\right)\right]\right\} \\
&= \max_{\hat{K}',n}\left\{\log\left(z\hat{K}^{\theta}n^{1-\theta} + (1-\delta)\hat{K} - (1+\gamma)\hat{K}'\right) - v(n) + \beta E\left[V\left(z',\hat{K}'\right)\right]\right\}
\end{aligned}$$

s .t. $\log z' = \frac{-\sigma^2}{2} + \rho\log z + \varepsilon'$

The FOCs are:

$$\hat{K}' : \frac{1+\gamma}{\hat{c}} = \beta E\left[V_2\left(z',\hat{K}'\right)\right]$$

$$n : \frac{(1-\theta)z\hat{K}^{\theta}n^{-\theta}}{\hat{c}} = v'(n)$$

The envelope condition is:

$$V_2(z,\hat{K}) = \frac{\theta z\hat{K}^{\theta-1}n^{1-\theta} + 1 - \delta}{\hat{c}}$$

Substituting the EC into the first FOC and rearranging terms gives:

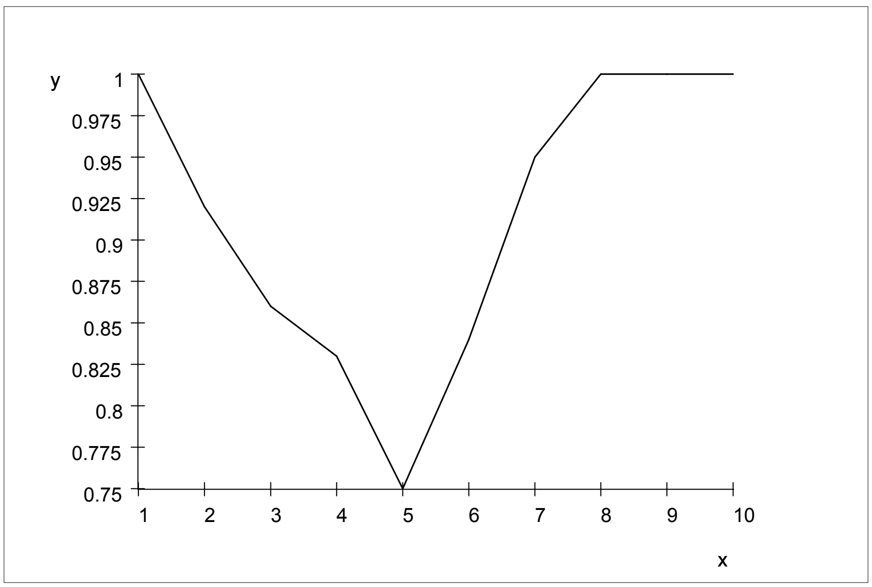
$$\frac{1+\gamma}{\hat{c}} = \beta E\left(\frac{\theta z' \hat{K}'^{\theta-1} n'^{1-\theta} + 1 - \delta}{\hat{c}'} \right)$$

$$v'(n) = \frac{(1-\theta) z \hat{K}^{\theta} n^{-\theta}}{\hat{c}}$$

These are necessary and sufficient conditions for a maximum because the opportunity set is convex and the objective function is concave in both arguments. Note that $v(.)$ is convex, hence $-v(.)$ is concave.

The planner's problem can be decentralized as a competitive equilibrium because the problem satisfies the conditions required by the Second Welfare Theorem: convexity of opportunity set, concavity of objective function, and no distortions of any sort.

(b) Time series for z_t:



To gain intuition, I will assume for while that $v(n)$ is linear (remember that a linear function is still convex) and that $\delta = 1$. In this case, we are able to obtain closed form solutions for the policy functions. Remember from Hansen's HW 4, Q3, that

$$n_t = \bar{n}$$

$$\hat{K}_{t+1} = \hat{I}_t = \left(\frac{\theta\beta}{1+\gamma\theta\beta} \right) z_t \hat{K}_t^{\theta} \bar{n}^{1-\theta}$$

$$c_t = \left(1 - \frac{\theta\beta}{1+\gamma\theta\beta} \right) z_t \hat{K}_t^{\theta} \bar{n}^{1-\theta}$$

$$Y_t = z_t \hat{K}_t^{\theta} \bar{n}^{1-\theta}$$

Taking logs gives:

$$\log n_t = \log \bar{n}$$
$$\log \hat{I}_t = \log\left(\frac{\theta\beta}{1+\gamma\theta\beta} \right) + \log z_t + \theta \log \hat{K}_t + (1-\theta)\log \bar{n}$$
$$\log \hat{c}_t = \log\left(1 - \frac{\theta\beta}{1+\gamma\theta\beta} \right) + \log z_t + \theta \log \hat{K}_t + (1-\theta)\log \bar{n}$$
$$\hat{Y}_t = \log z_t + \theta \log \hat{K}_t + (1-\theta)\log \bar{n}$$

It is straightforward to see from that in this case all variables, except labor, there is positive relationship between the technology shock and all variables, except labor. In another words, these variables are procyclical. Moreover, the size of the change is one-to-one. Now, lets go back to the general model. The stochastic system of equations is given by:

$$1+\gamma = \beta E_t\left[\left(\frac{\hat{c}_t}{\hat{c}_{t+1}} \right)\left(\theta \frac{\hat{Y}_{t+1}}{\hat{K}_{t+1}} + 1 - \delta \right) \right]$$
$$v'(n_t) n_t^{\theta} = \frac{(1-\theta) z_t \hat{K}_t^{\theta}}{\hat{c}_t}$$
$$\hat{c}_t + \hat{I}_t = \hat{Y}_t$$
$$\hat{I}_t = (1+\gamma)\hat{K}_{t+1} - (1-\delta)\hat{K}_t$$
$$\hat{Y}_t = z_t \hat{K}_t^{\theta} \hat{n}_t^{1-\theta}$$

In the general model, investment becomes less volatile than in the example above because of the undepreciated part of the capital, that is, investment no longer fully adjusts to current shocks as it used to do in the example above. Additionally, labor will no longer be constant but will change with the technology shock. In fact, labor has to fall because the marginal product of labor falls with the shock. My guess is that (1) output will fall more than labor, and that (2) consumption will fall more than investment. To see the first claim, take the log of the FOC for labor:

$$\log v'(n_t) + \theta \log n_t = \log(1-\theta) + \log \hat{z}_t + \theta \log \hat{K}_t - \log \hat{c}_t$$

In period t, the stock of capital is given (it was chosen in the previous period). Hence, a fall in z_t will reduce the left-hand side of the above equation in the proportion of one-to-one. However, consumption is also falling, acting to counterbalance the fall in z_t. The effect of consumption is equivalent to a negative income effect on the labor supply, meaning that workers will try to work a little bit more to compensate for the income loss. Anyway, we know that overall labor has to fall because a decrease in z_t reduces the marginal productivity of labor, which discourages labor demand. How much labor falls depend on the shape of the function $v(n_t)$. Output falls more than labor because z_t and n_t are falling at the same time. To see this clearly just take the log of the production function and remember that the current stock of capital is given:

$$\log \hat{Y}_t = \log z_t + \theta \log \hat{K}_t + (1-\theta)\log n_t$$

Try to show claim (2), using the euler equation:

$$\frac{1+\gamma}{\hat{c}_t} = \beta E_t\left(\frac{\hat{R}_{t+1}}{\hat{c}_{t+1}}\right)$$

where $\hat{R}_{t+1} = \theta \hat{Y}_{t+1}/\hat{K}_{t+1} + 1 - \delta$ is tomorrow's return on capital.

(c) I think there is a typo in the question, $v(n)$ should be written without the negative sign because there is already a negative sign in the objective function. Anyway, remember that a linear $v(n)$ corresponds to the indivisible labor model, which delivers large variations in the labor supply. Therefore, my guess is that linear preferences overs leisure implies more volatility of the labor supply. One way to see this is to consider the log of the FOC for labor again:

$$v(n_t) = \phi n_t \ :$$

$$\log \phi + \theta \log n_t = \log(1-\theta) + \log \hat{z}_t + \theta \log \hat{K}_t - \log \hat{c}_t$$
$$\log n_t = \frac{1}{\theta}\log \hat{z}_t + \ \ldots$$

$$v(n_t) = \phi n_t^3 \ :$$

$$\log(3\eta n_t^2) + \theta \log n_t = \log(1-\theta) + \log \hat{z}_t + \theta \log \hat{K}_t - \log \hat{c}_t$$
$$\log n_t = \frac{1}{2+\theta}\log \hat{z}_t + \ \ldots$$

Therefore, the effect of a decrease in z is smaller in the second case, ceteris paribus.

(d) This is abnormal. In fact, the pattern displayed by z in the previous figure does no reflect variations along a normal business cycles. It resembles more a depression (I guess this is Lee Ohanian's question, meaning that the numbers for z are actual numbers computed for the US Great Depression). In order to investigate this phenomenon I would try to see what was going on in the main markets: labor market, and capital markets. Cole and Ohanian argue that the cartelization of the labor market by Roosevelt's policies kept wages artificially high and contributed to the persistence of the Great Depression. Others, including Irving Fisher (1933), argue that the problem was in the financial system (half of the banks failed during the depression, and financial intermediation collapsed). Others argue that the problem was a wrong monetary policy (too tight money supply, which reduced liquidity). Anyway, the persistence of the Great Depression was probably caused by some big distortion (which nobody fully understands yet) either in the labor market, or in the capital markets, or both.

SPRING 2004 - QUESTION 2

(a)

Bellman equation for the Planner's problem:

$$v(k) = \max_{k',n}\{\log(k^\alpha n^{1-\alpha} - k') - n + \beta v(k')\}$$

A sequence-of-markets equilibrium for this economy is an allocation $\{c_t, k_t, n_t\}_{t=0}^{\infty}$ and prices $\{w_t, r_t\}_{t=0}^{\infty}$ such that

(i) Given prices, consumers solve:

$$\max_{\{c_t, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t[\log(c_t) - n_t]$$

such that

$$c_t + k_{t+1} = w_t n_t + r_t k_t, \quad \forall t$$

k_0 is given

(i) Given prices, firms solve:

$$\max_{\{K_t, N_t\}}\{K_t^\alpha N_t^{1-\alpha} - w_t N_t - r_t K_t\}, \quad \forall t$$

Note: I'm using capital letters to denote firms' choice variables.

(iii) Markets clear:

$$n_t = N_t, \forall t$$
$$k_t = K_t, \forall t$$
$$c_t + k_{t+1} = k_t^\alpha n_t^{1-\alpha}, \forall t$$

To find the steady state, use the Bellman equation, take FOC, envelope condition, and find the euler equations:

$$\text{FOC } k' : \frac{1}{k^\alpha n^{1-\alpha} - k'} = \beta v'(k') \quad (1)$$

$$\text{FOC } n' : \frac{(1-\alpha)k^\alpha n^{-\alpha}}{k^\alpha n^{1-\alpha} - k'} = 1 \quad (2)$$

$$\text{EC } k : v(k') = \frac{\alpha k^{\alpha-1} n^{1-\alpha}}{k^\alpha n^{1-\alpha} - k'} \quad (3)$$

Note: the maximization problem is well defined and there is no corner solution because $n \in (0,1)$ by assumption.

Two equations describe the planner's solution, the capital euler equation (combining 1 and 2) and the labor euler equation (2), respectively:

$$\frac{1}{k^\alpha n^{1-\alpha} - k'} = \beta \frac{\alpha k'^{\alpha-1} n'^{1-\alpha}}{k'^\alpha n'^{1-\alpha} - k''}$$
$$\frac{(1-\alpha)k^\alpha n^{-\alpha}}{k^\alpha n^{1-\alpha} - k'} = 1$$

In steady state we have:

$$1 = \beta\alpha \bar{k}^{\alpha-1}\bar{n}^{1-\alpha}$$
$$(1-\alpha)\bar{k}^\alpha \bar{n}^{-\alpha} = \bar{k}^\alpha \bar{n}^{1-\alpha} - \bar{k}$$

Solving the first equation we get:

$$\bar{k} = (\alpha\beta)^{\frac{1}{1-\alpha}}\bar{n}$$

Substituting this into the second equation gives:

$$(1-\alpha)(\alpha\beta)^{\frac{\alpha}{1-\alpha}} = (\alpha\beta)^{\frac{\alpha}{1-\alpha}}\bar{n} - (\alpha\beta)^{\frac{1}{1-\alpha}}\bar{n}$$

Solving for $\bar{n}$:

$$\bar{n} = \frac{1-\alpha}{1-\alpha\beta}$$

Substituting this into the equation for $\bar{k}$ gives:

$$\bar{k} = (\alpha\beta)^{\frac{1}{1-\alpha}}\left(\frac{1-\alpha}{1-\alpha\beta}\right)$$

(b) Bellman equation for the Planner's problem:

$$v(k,e) = \max_{k',n}\left\{\log\left(k^\alpha (en)^{1-\alpha} - k'\right) - n + \beta v(k',e')\right\}$$
$$\text{s.t. } e' = (1-\delta)e + n^2$$

If you solve this problem (for example, using value function iteration), you will find the following laws of motion for labor, capital, and consumption:

$$n_t = \frac{1-\alpha}{1-\alpha\beta}$$
$$k_{t+1} = \alpha\beta k_t^\alpha (e_t n_t)^{1-\alpha}$$
$$c_t = (1-\alpha\beta)k_t^\alpha (e_t n_t)^{1-\alpha}$$

This allocation will be the same across all people.

In the case of indivisible labor, $n_t = \{0,1\}$, this is still longer true because the the coefficient of risk aversion is $\sigma = 1$.

Let π_t be the probability of working full time, that is, $\pi_t = \Pr(n_t = 1)$. Let c_{1t} be the consumption when working full time, and let c_{2t} be the consumption when unemployed. Expected utility of an individual consumer becomes:

$$\begin{aligned} u(c_{1t}, c_{2t}, \pi_t) &= \pi_t(\log c_{1t} - 1) + (1-\pi_t)(\log c_{2t} - 0) \\ &= \pi_t \log c_{1t} + (1-\pi_t)\log c_{2t} - \pi_t \end{aligned}$$

Remember from Hansen's Problem Set 4 that by introducing lotteries we "convexify" the problem and make decentralization possible. The (decentralized) problem of a consumer is:

$$\max \sum_{t=0}^{\infty} \beta^t [\pi_t \log c_{1t} + (1-\pi_t)\log c_{2t} - \pi_t]$$

such that

$$\begin{aligned} \pi_t c_{1t} + (1-\pi_t)c_{2t} + k_{t+1} &= r_t k_t + w_t[\pi_t e_t 1 + (1-\pi_t)e_t 0] = r_t k_t + w_t \pi_t e_t \\ e_{t+1} &= (1-\delta)e_t + \pi_t^2 \end{aligned}$$

If you set up the Lagrangean problem (or the Bellman equation for the consumer) and take FOC with respect to c_{1t} and c_{2t} you will conclude that:

$$c_{1t} = c_{2t} \equiv c_t$$

Therefore, regardless the individual state (employed, unemployed), each consumer will consume the same amount, as in the previous problem. This is because preferences are separable in consumption and leisure (and also because the labor parameter in the utility function is normalized to one). For a more general class of utility functions, for example, $\frac{1}{1-\sigma}(c_t^{\theta}(1-n_t)^{1-\theta} - 1)$ for some $\theta \in (0,1)$, this result no longer holds, since in this case agents would consume more when employed. As for the labor supply, not the same people will be working in every period: they work with probability π_t and don't work with probability $1-\pi_t$, although they still can have a flat consumption as I said before.